Planet Maths

A numerical tour for everyone

Everything is mathematical

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Miquel Albertí

Everything is mathematical

To Pilar

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Preface

All cultures are characterised by their own beliefs and rituals, a philosophy of life, a method of social organisation, a language, a literature, cuisine, artistic styles, a system of trade, technology, archaeology and, of course, by mathematics.

Our mathematics is a product of Western culture, which emerges on a daily basis at specific sites such as universities and in research centres. However, mathematics has also been developed in other areas, professional or otherwise, the knowledge of which has developed on the sidelines of Western academic traditions.

The history of mathematics as we know it is a Western one. Even so, its foundations lie before our culture and before academia. Anthropological research has paid little attention to mathematics. Generally speaking, it has not gone beyond anecdotal records of numbering and counting systems. Nor did Western colonists worry too much about indigenous mathematical ideas, which thus remained hidden and were considered to have developed only though craftwork and other artisan cultural activities.

We are not referring to a sort of primitive folk mathematics, but to mathematics that has arisen independently, with specific goals to solve practical problems. All cultures require things to be done with quality, rigour and precision. All cultures count, measure, locate and design. Ethnomathematics is the vernacular mathematics developed by peoples and cultures to meet their particular needs. If these types of activities are carried out everywhere, it is logical to think that they hold clues to the roots of a culture's mathematics. Clearly we should not expect to find mathematics as we know it in an academic context. We should expect raw mathematics, unpolished or unsculpted with respect to the academic perspective, ideas based more on experience and practice than on proof. However, it does not lack the necessary logic.

Broadly speaking, ethnomathematical research aims to reveal the indigenous mathematics of peoples and cultures, to evaluate its use and incorporate it into the body of formal mathematics so it can be developed further and used as an educational resource. Where can we find ethnomathematics? How can we find it? What can we do with it?

This book is a mathematical journey across the world through time and cultures. We shall see that various cultures have developed their own numbering systems and indigenous mechanisms for calculating. One fruit of this labour is the ancestors of calculators: Inca quipus and the Chinese abacus.

Spatial, two-dimensional and three-dimensional organisation is fundamental to architecture and ornamentation. Its mathematical importance is crucial for creating patterns, both figurative and for reproduction. This is true to such an extent that a people or culture can be easily recognised by a geometric design. From prehistoric times to the present day, and all across the globe, symmetry is a universal paradigm of cultural expression.

Games play a fundamental role, since they commit us to the acceptance, knowledge and monitoring of the series of rules that constitute them. Herein lies the logic of games and the basis of the justification of their results and events. It is in games that a culture expresses its way of understanding chance.

Not all cultures separate mathematics from other aspects of culture. In a ritual or ceremony, uninitiated eyes may interpret acts as theatre, dance, music or geometry, but for those who carry them out, they are all the same thing. However, this is not the goal. We will not discuss here whether a native person believes they are practising mathematics, but we shall concern ourselves with their mathematics from our perspective.

In the end, we shall find the answers to the questions posed above in ethnomathematical knowledge. We are a mathematical species.

It is not possible to discover the mathematics that arises in certain arenas without turning to those who practise it. For this reason, I would like to express my gratitude to Ibu Ketut for her cooperation with the material on the preparation of offerings on the island of Bali (Indonesia). A special thank you to Kamini Dandapani, from Chennai (Tamil Nadu, India), whose photographs illustrate and explain the underlying mathematical ideas of the kolams. Support from Dolors Guixà and Joan Serra, L'art ORL Vitrall (Sabadell, Spain) was crucial in the section about the mathematical links to glass-making. I would also like to express my gratitude to all of the above for helping illuminate mathematical ideas and activities that often remain hidden.

Chapter 1

Ethnic Origins of Mathematics

Where there is culture, there is mathematics

The word mathematics is used throughout the world. It reflects the way the subject is studied everywhere using practically identical methods and content. At schools, institutes and universities throughout the world, students are taught and learn calculation, Thales' and Pythagoras' theorems are explained, problems are solved using equations and systems, and mathematical models are developed for the most wide-ranging phenomena. This conception of mathematics also covers its applications in other disciplines, regardless of whether they are related to the sciences.

Mathematical methods make use of increasingly sophisticated tools. If the Platonic, constructivist perspective was developed under the limitations of the ruler and the compass, today it is impossible to understand the development of mathematics without making reference to the latest technology – from the calculator all the way to the most sophisticated computer software.

This spectrum illustrates the universal nature of mathematics. However this universal nature is, above all, institutional. It is developed in institutions and coordinated using educational projects in different countries. Subtleties aside, the mathematics that is taught and learnt, both in the East and West, North and South, is essentially the same.

However, another universal aspect of mathematics is rooted in all of the world's cultures, because the development of mathematical ideas and methods for solving problems is also universal. From this perspective, mathematics is a pan-cultural phenomenon. Alan Bishop was first to suggest this idea in 1991. Thanks to his book *Mathematical Enculturation*, we are all more aware of the role mathematics plays as a cultural artefact and fundamental element in the spread of culture.

The stereotypical image of the cultured person who freely admits to know nothing of mathematics should now be a thing of the past. The very idea of culture

implies many contexts in which the mathematician is present in the background. Can a people or culture exist or understand itself without mathematics? The answer is an emphatic no.

Culture is the collection of knowledge developed by people over the course of time in a way that facilitates and so characterises their way of life. Isolated groups of people may develop different cultures, the differences of which may appear in the ties of cohesion between communities, housing, eating habits, the work they do to survive, beliefs, myths, fears, etc. With the passing of time, each culture will come to develop systems of social and political organisation, a language, a philosophy of life and existence, rituals and beliefs, technology and cultural manifestations of the most diverse nature: music, dance, ornamentation etc.

All this has occurred at different times in different places. The predominant Western culture has only been aware of this for a few hundred years. Prior to the 15th century, nothing was known about the American continent and hardly anything was known about what lay beyond Europe. Of the land beyond India, almost all that was known was the information the explorer Marco Polo had recounted upon his return from his trip to Cathay, now better known as China – and there is no evidence that he even went there! There was no knowledge of the Pacific Ocean and its island cultures. The first name given to Australia, the island continent, was *Terra Incognita*.

However, for millennia, these lands that were unknown to the Europeans were inhabited by people who had developed their own systems of knowledge. They communicated using their own languages and some had symbols for writing. They lived in houses built using tools and utensils, which they used to transform the materials around them: wood, bamboo, mud, leaves etc. They spent the time playing with pebbles that were distributed in an orderly fashion into hollows chiselled into wooden boards. They travelled and traded with their neighbours over land and sea.

These peoples knew how to do things. Nobody would doubt that they knew how to hunt, organise themselves, build, cook, sail, select, locate, design, speak, write or play. They also counted, calculated and measured. However, if each culture is able to develop its own cultural artefacts that differ from those of other people, such as its beliefs, philosophy of life, architecture and exchange system for trading or works of art, might the same thing not happen with mathematics?

The vernacular mathematics that cultures or groups of people are able to develop is known as ethnomathematics. The term was coined by the Brazilian mathematician and teacher Ubiratan d'Ambrosio, towards the end of the 1980s. There have been

and still are many distinct cultures in the world. Their specific mathematical ideas allow us to talk of an ethnomathematical world.

THE FOUNDING FIGURES OF ETHNOMATHEMATICS

The relationship between mathematics and culture dates back to the first anthropological studies, such as Gay and Cole's work on the Kpelle people in Liberia, Africa. However, the development of the body of knowledge that is now known as ethnomathematics has been attributed to Alan Bishop (from the United Kingdom) and Ubiratan D'Ambrosio (Brazil), although the work of Paulus Gerdes (Mozambique), Marcia Ascher (United States of America) and Claudia Zaslavsky (United States of America) are also important and have been inspirational in the discipline's development.

Ubiratan D'Ambrosio.

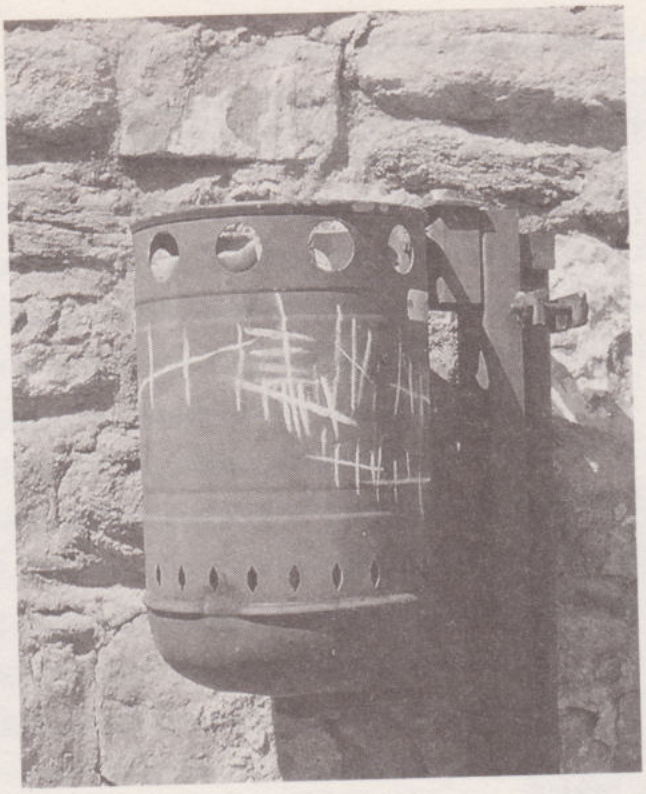
D'Ambrosio was born in São Paulo and obtained his degree and PhD in mathematics at the city's university. He completed postdoctoral research at the Department of mathematics at Brown University, in the United States.

Alan Bishop is now emeritus professor of education at the faculty of education at Monash University in Australia. However, he began his professional life at Cambridge (England). He advises UNESCO on matters related to mathematical, technical and scientific education.

The origins of mathematics as we know it in our culture date back thousands of years. Just like the totality of our culture in its own right, it was developed with ideas from different places and different peoples. We have inherited our form of thinking from the Sumerians, the ancient Egyptians and Greeks, the Arabian and Indian worlds, and from China. The same can be said of academic mathematics, whose origins are, in fact, entirely ethnomathematical. Our mathematics is the product of ancient cultural exchanges scattered across the planet. Nothing began at the same time or in the same place.

All we need to do is leave our house to see that mathematical activity takes place everywhere and involves elements that do not always belong to the academic world.

The cultural content of following images may give rise to many commentaries, some of which will be mathematical in nature.



A bin located in the district of Morella, Castellón (photograph: MAP).

The image shows a stone wall, to which is attached a metal object we recognise as a bin. It is cylindrical with a bulge at the bottom. It has two series of perforations along its perimeter, which serve as ornamentation. The holes of the upper part are circular, whereas those of the lower part are hexagonal. They are arranged in a pattern of regular intervals, with a rhythm or proportion of two hexagons per circle. Some inscriptions have also been made in chalk. These are seven series of four parallel lines crossed by another horizontal line.

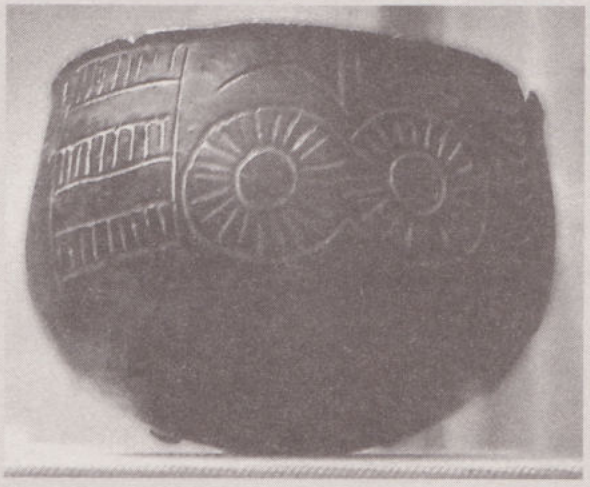
From a visual inspection of this object, it is possible to venture a hypothesis. One is that a culture able to manufacture an object such as this has technology that allows it to work with refined metals, to mould them and perforate them in a controlled manner, according to predetermined shapes and patterns. The object does not appear to be hand made, but mechanically manufactured, guaranteeing its reproduction in exact copies. The inscriptions, on the other hand, appear to have been made by hand. Whoever made them must have been able to count to five at least seven times, giving a total of 35 units. We will never know what they were counting. There is also the possibility that they were not for counting but were a record of an unconscious

rhythm, such as when we tap our feet repeatedly on the ground without counting the tempos or the bars of the music we are listening to. In such situations, we are merely marking the rhythm.

At any rate, these statements are based on a cultural tuning. We recognise the object as a bin. However, who are we? The inhabitants of Morella, in Castellón, where the photograph was taken? The inhabitants of Spain? Europeans? Would a tuareg from Mali, a Sami from Lapland or a rice farmer on the island of Luzon in the Philippines recognise it as a bin? Perhaps not. What they would most surely recognise would be the metallic nature of the object, its cylindrical shape and its circular and hexagonal perforations. They would also know how to count the numbers of the different types of holes, although it is possible their terms and counts would be different from ours. Perhaps this would be because they have not learnt them at school, but from their ancestors.

CONCENTRIC CIRCLES 6,000 YEARS AGO

The prehistoric settlement of Los Millares in Almería, Spain, is a Copper-Age archaeological site of a culture that inhabited the southern part of the Iberian Peninsula. Ceramic artefacts discovered in the region exhibit geometric ornamentation, such as this bowl with concentric circular lines in the shape of eyes and various series of parallel and equidistant lines. The eyes appear to be the symbol of this people, since the design is present in the majority of the remains discovered at the site.



A bowl from Los Millares, Almeria (photograph: José Manuel Benito Álvarez).

The next photograph requires an entirely different investigation. The image was taken at the entrance to a cave house of the ancient Galera people in the Granada region of Spain. Our culture recognises these symbols as numbers. And in spite of not having an operator symbol, the layout corresponds to the method of manual multiplication we all learn at school. The fact it is a multiplication is confirmed by the fact that if we calculate the answer, multiplying 150 by 12, the result is 1,800.

Entrance to a cave house in Galera, Granada (photograph: MAP).

And what can we say about this other photograph of a wall of the Catalonia Plaza Hotel in Barcelona, Spain?

Looking at the image may lead us to suggest the hypothesis that the tiles that make up the wall were designed based on a celebrates mathematical relationship, since the square formed by each window can be broken down into the sum of two smaller squares and two equal rectangles. If a is the side of the smallest square and b is the side of the medium square, the rectangles have dimensions $a \times b$, and the whole window is a square with sides $a + b$. Hence:

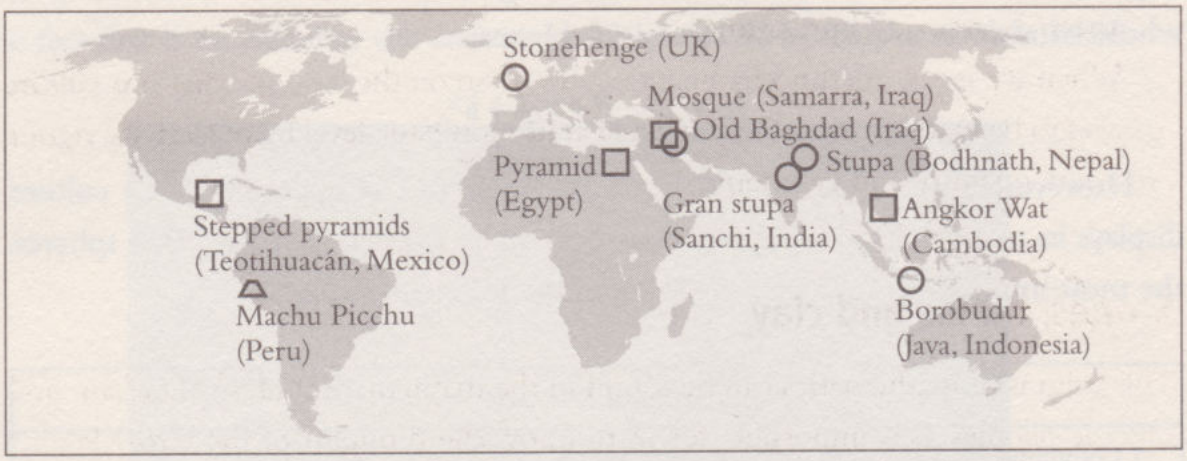
$$(a+b)^2 = a^2 + 2ab + b^2.$$

However, it is not just in areas such as design and architecture that a culture displays its mathematical ideas. Mathematics can be explicit in many other spheres, the most important of which are shown in the following table:

Cultural artefacts	
1 Communication	Language, writing, symbols...
2 Beliefs	Philosophy, cosmology, religion, rituals, the interpretation of dreams...
3 Environment	Location, fauna, flora, geology.
4 Work	Agriculture, livestock, hunting, fishing...
5 Technology	Tools, handicrafts, arms, systems of units...
6 Architecture	Houses, places of worship, tombs, urban development...
7 Food	Food, drink, cooking...
8 Attire	Clothes, accessories...
9 Exchange	Trade, economics, market, currencies...
10 Art	Music, dance, literature, painting, sculpture...
11 Leisure	Games, betting, sports...
12 Relationships	Social, kinship...

Many cultural practices involve implicit mathematical ideas, often hidden, such as those classified by the Mozambican teacher Paulus Gerdes. Revealing these ideas allows us to understand the mathematics of a people or culture. However, alongside this hidden mathematics, there may also be more evident mathematical ideas, identifiable in the thought processes the authors of these practices use when they are executed. We shall see that some of these are inseparable from the culture in which they are developed.

We must follow various paths to identify the ethnomathematics of a culture. Given that some features of mathematics are objectivity, rigour and precision, both in terms of quantity and form, studying cultural practices or artefacts in which these aspects are present allows us to uncover vernacular mathematical ideas.



Some of the great architectural constructions throughout the world based on the circle, the square and the trapezium.

A first approximation leads to architecture, craft, technology, commerce and games. Focusing on the practical activities required to execute them, Alan Bishop specifies six universal aspects of mathematical activity present in all cultures: counting, measuring, locating, designing, playing and explaining. Hence, where there is counting, measuring, locating, designing, playing or explaining, it is plausible those who carry out these activities are putting their own mathematical ideas, at least those indigenous to their culture, into practice. To understand them is to understand their ethnomathematics.

Is ethnomathematics a worthwhile pursuit? Or is it nothing more than curious anecdotes that can be used to illustrate an exotic journey around the world. The answer to this question involves a number of major considerations. We shall see that some vernacular mathematical ideas not only facilitate solving traditional mathematical problems but also improve the solution in the cultural context in which they are developed. And that creates a clearer understanding of other mathematical ideas from the academic realm. Furthermore, we must bear in mind that ethnomathematics has not enjoyed the treatment given to academic mathematics. As Gerdes and others have already observed, Western colonialism is partly responsible for the silence of a large part of ethnomathematics and the difficulties it has faced in its development.

There is no reason for our conception of what is mathematical to coincide with what a native Navajo, Shuar or Maori regards as mathematical. It is

possible that these cultures lack a classification for what is mathematical and, where such a classification exists, it may be the case that its features do not correspond precisely to those attributed to our own. The same thing happens with other cultural manifestations. Dances can be regarded by some people as a prayer, a supplication or a gesture of gratitude to a divine being, while others see them as artistic expression.

When we speak of ethnomathematics, we do so on the basis of what our culture believes to be mathematical, characterised at its most basic level by objectivity, rigour, precision, quantity and geometry.

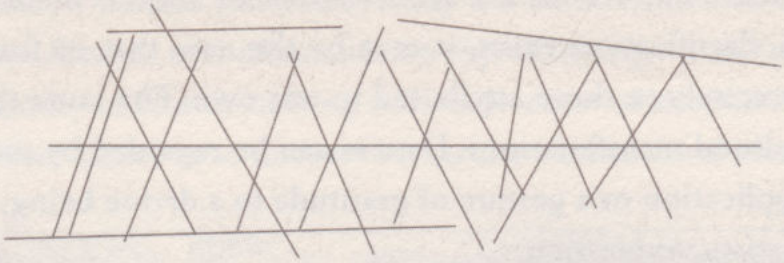
Stones, bones and clay

The origins of mathematics can be found in the mathematical ideas of certain prehistoric peoples. It is impossible for us to know the thoughts of the Cro-Magnon or Neanderthal man or their ancestors. We can only speculate at what mathematical ideas they might have had in their head using the traces of their existence that have survived the passing of time.

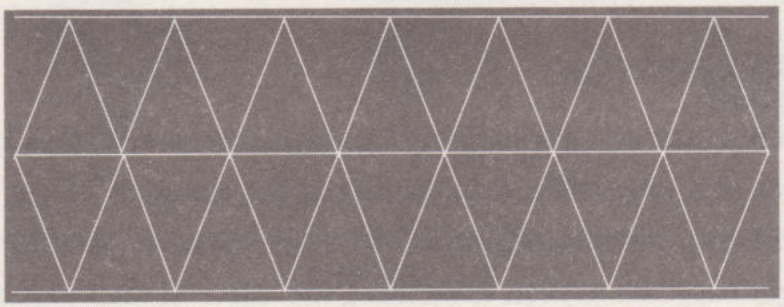
In 2003, in the Blombos Cave in South Africa, a fragment of ochre believed to be around 72,000 years old was found. It had been worked into a geometric design, shown in the photograph below:

A petroglyph from the Blombos Cave, South Africa (photograph: Chenshilwood).

The design is 60 mm long with a maximum width of 2 mm. Its geometric nature arises from the two series of triangles of which it is made up, which are the result of drawing a series of parallel lines. Reproducing the design on the surface allows us to better appreciate its geometric nature:



Perhaps the irregular surface of the stone and insufficiently developed technology prevented the creator from developing what a modern observer would describe as a triangular grid with greater rigour and precision.



Based on the line work of the design, they would say the triangles were not drawn separately, but appear as a consequence of the intersection of three sets of parallel line segments. The first is made up of three parallel horizontal segments; the second of eight parallel segments sloping towards the left; and the third of nine parallel segments sloping towards the right.

We will never know if the person behind the design had the concepts of a straight line, segment, angle, parallel and symmetry in their head. Nor will we know if the carving was an emblem or sign of something or someone, if it had a practical use, or if it was produced purely as an interesting shape. However, we must conclude that, consciously or unconsciously, the goal was to execute a work encompassing these mathematical ideas. The unfathomable mysteries of the reality and the absence of appropriate technology held the creator back. At any rate, we are dealing with a prehistoric relic from which it is possible to infer the existence of mathematical thought.

Much more recent is the large baboon bone found in the area of Ishango in 1960 in what was then the Belgian Congo, now the Democratic Republic of Congo. It is estimated to be some 20,000 years old. At first it was believed the bone served as a stick for counting, since it has a series of grooves carved at regular intervals, as shown in the image:

Two views of the Ishango bone (the Brussels Science Museum).

The bone contains three columns of marks grouped as follows:

Column A: $11+13+17+19=60$.
Column B: $3+6+4+8+10+5+5+7=48$.
Column C: $11+21+19+9=60$.

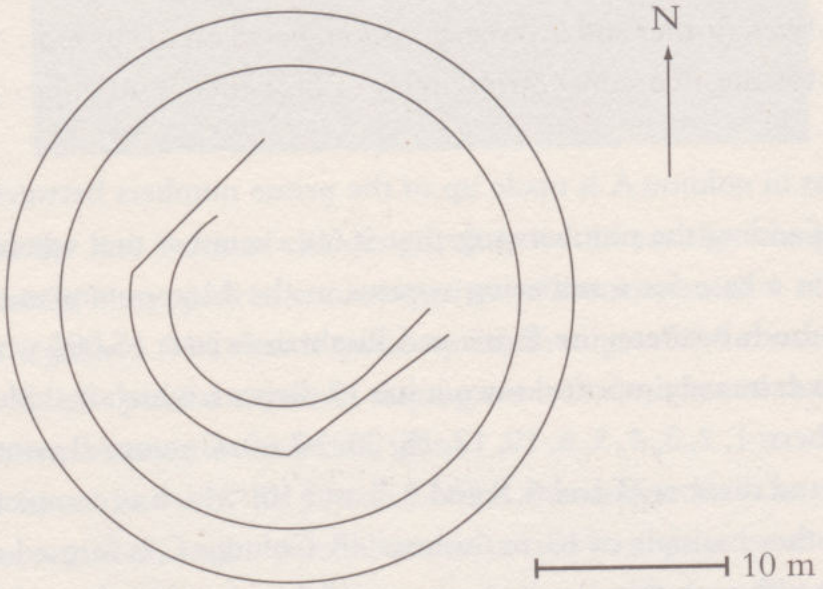
The series in column A is made up of the prime numbers between 10 and 20. The result of adding the numbers together is 60, a number that would be of great importance as a base for numbering systems in the Mesopotamian cultures that grew up the lands between the Tigris and Euphrates rivers 15,000 years later. The number 60 is extremely practical since it has 12 divisors, which include the first six natural numbers: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60. Column B contains a series of multiples and divisors (3 and 6, 4 and 8, 5 and 10) which is completed with the 7 to give another multiple of 12, in this case 48. Column C is formed of a series of odd numbers, although these are not prime, and also gives 60 when added together.

Is it a coincidence that the numbers in the three columns of grooves add up to 60, 48 and 60? Does this mean that those who made the marks were already aware of the ideas of multiplication and division, clearly present in the pairs 3 and 6, 4 and 8, and 5 and 10? Can we conclude from this that they also had an idea of indivisible or prime numbers, such as 3, 5, 7, 11, 13 and 19? It is hard to answer these questions, especially when we bear in mind that the marks on the bone do not all have the same length and that some are discontinuous. Do the broken lines refer to one or two units? Or are they errors occurred when making the cut?

The most plausible mathematical phenomenon that can be deduced from the marks of the Ishango bone is the establishment of a 1-to-1 correspondence between the marks and the other objects, which constitutes the basis of a count.

This is a substantial difference from the petroglyph of the Blombos Cave in South Africa. The marks on the Ishango bone do not appear to be geometric in nature but numeric. In contrast, the spirit of the Blombos design is the opposite – geometric not numeric.

Many years after the South African petroglyph and the bone from the Congo, we can find a structure on the European continent that refers to numbers and geometry. The structure in question is the megalithic ruins of Stonehenge, on England's Salisbury Plain. The structure of Stonehenge is circular and is made up of four concentric circles constituted of menhirs of varying heights. It is regarded as the most sophisticated megalithic work, since it involves the erection of the menhirs and the composition of dolmens to create a larger work.



The concentric circular plan view of Stonehenge.

The outer circumference of the monument, which has a diameter of 30 m, is made up of enormous stones in the shape of right-angled prisms, the upper parts of which were originally closed by lintels. Inside this circle is another made up of smaller blocks, which enclose a structure in the shape of an open horseshoe. This structure contains the slab referred to as the altar. The monument is surrounded by a circular ditch with a diameter of just over 100 m and was erected around 2500 BC, although the oldest part dates back to 3100 BC.

The purpose of the construction is unknown, however three of the most probable functions attributed to it are as a place of worship, a funerary monument or an

astronomical observatory. In this respect, it is worth noting that on the summer solstice, the Sun rose in line with the central axis of the building. At sunset on the same day, the Sun disappeared in line with the axis of what is called Woodhenge, a place close to Stonehenge in which the bones of many animals and other objects have been found, possibly indicating the celebration of religious ceremonies or ancestral worship.

The geometry of Stonehenge is different from the geometries we have considered thus far because it is circular. However, it has something in common with the previous examples, because its structure is based on a series of repetitions in a fixed pattern that gives the construction its character. The Blombos petroglyph is based on repeated triangles; the Ishango bone is based on equidistant markings. In this case, the circle is repeated, with the repetition preserving the powerful structural order of having the same centre.

We can go even further and make assumptions based on the proportion between the diameters of the two concentric circles of Stonehenge of approximately 30 m and 24 m:

$$\frac{30\text{ m}}{24\text{ m}} = \frac{5}{4} = 1.25.$$

However, the measurements of these circles may well have been 30.4 m and 24.1 m. In this case, the proportion would be:

$$\frac{30.4\text{ m}}{24.1\text{ m}} \approx 1.26 \approx \sqrt[3]{2} = 1.259921...$$

Given that 1.26 is an excellent approximation of the cube root of 2, should we perhaps conclude that those who erected Stonehenge were aware of proportionality and that they drew these concentric circles based on the cubic root of 2? The answer to this question has to be no because there is no evidence to support this hypothesis.

There are three aspects of this megalithic monument worth pointing out. Its geometric structure is based on concentric circles; it is related to astronomy, and it appears to have been created by a culture that valued geometric rigour.

Long before Stonehenge was built, in the land between the Tigris and Euphrates, the Babylonian culture recorded its thoughts on clay tablets. However, in spite of being petroglyphs and being geometric in nature, the signs marked on these clay tablets can be classified as writing. Hence many of the things we know about the

peoples that inhabited Mesopotamia are not just hypotheses and intuitions, but interpretations of their writing.

In the same area, some five thousand years ago, the Sumerian culture began to use ideograms to write down its language. With the passing of time, these were refined until some 1,000 years later they had transformed into the characters now called cuneiform. This style of writing was adopted by other peoples and gave rise to the ancient Persian alphabet.

Some 2,000 characters of this system of writing are known, although in later periods the number reduced to 600. The following are the characters for writing the first 59 natural numbers and shows the use of base 10 in the Babylonian decimal numbering system:

Symbols of the Babylonian sexagesimal numbering system.

However, the Babylonian numbering system was not just limited to the decimal base. The YBC 729 tablet is a small tablet containing a square and its two diagonals. On this tablet and others, it becomes clear that the Babylonians made use of numbers for much more than counting.

The YBC 729 Babylonian clay tablet.

The numbers accompanying the drawing may refer to the length of the segment on which they were written. However, the numbers 42, 25 and 35 appear to be far moved from the side and the diagonal. What is the relationship between 30, 1, 24, 51, 10, 42, 25 and 35? Where have these numbers come from?

Imagine that the length of side c is 30 units. Let us calculate the length of its diagonal D:

$$D = 30 \cdot \sqrt{2} = 42.4264068...$$

This gives us one of the figures, 42. However, the Babylonians used the sexagesimal numbering system. Let us transform the result into this system. This can be done automatically using a calculator:

$$30 \cdot \sqrt{2} \rightarrow 42^\circ\ 25'\ 35.06''.$$

Hence, 42, 25 and 35. It would not appear to be an exaggeration to say that whoever engraved this tablet, or whoever commissioned its creation, was thinking of the calculation of the diagonal of the square of 30 units and that they wrote the result of their excellent approximation in sexagesimal notation: 42° 25' 35".

However, we still have to explain the numbers 1, 24, 51 and 10. Could this be the quotient, the proportion between the diagonal and the side of the square? Let us calculate this proportion in sexagesimal notation:

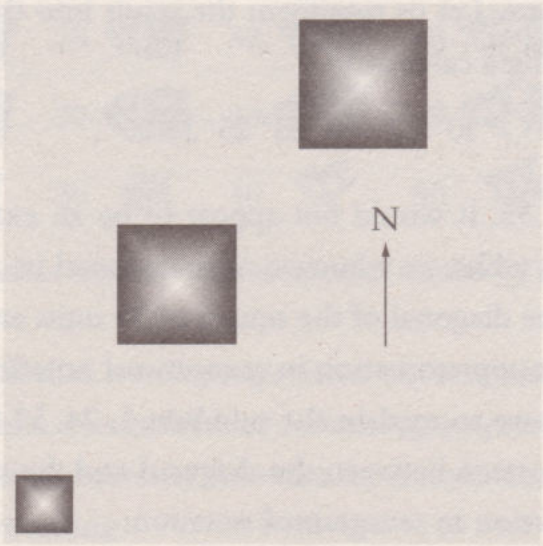
$$\frac{D}{c} = \sqrt{2} \rightarrow 1^\circ\ 24'\ 51.17''.$$

Hence, in sexagesimal notation, the number engraved above the diagonal is an extraordinary approximation to the square root of two. This confirms the hypothesis that the Babylonians had geometric knowledge and that, although they did not have a proof of Pythagoras' theorem, they knew how to calculate the diagonal of a square. What cannot be deduced from the tablet is how they arrived at these results. According to the content of another tablet, Plimpton 322, they were aware of Pythagorean triples and calculated the proportions between them, or what could have been the sides of right-angled triangles, now known as trigonometric ratios (tangent or cosecant). However, this does not guarantee that they were aware of Pythagoras' theorem and much less of its proof. How then did they obtain the above results? Did they perhaps use an iterative procedure to converge on such a precise approximation to the square root of 2?

One problem with the Babylonian numbering system was the absence of a symbol for zero. How is it possible to distinguish between the numbers 106 and 16 without a zero? To start with, a blank space was left. However, the problem remained. How was it possible to distinguish between three blank spaces (e.g. 10,006) and two (e.g. 1,006)? The Babylonians solved the problem by making use of separator symbols, although this made calculations much more difficult.

Pyramids and papyrus

A millennium and a half before Stonehenge, and almost a millennium before the clay tablets and their cuneiform writing, pyramids were being built in Egypt:



Orientation of the pyramids of Giza.

It is possible that we will never know the answers to some questions regarding their construction. However, their shape, orientation and dimensions suggest that mathematical ideas were involved. The pyramids were mausoleums in which the governing pharaoh, who enjoyed total and absolute power over his people, was laid to rest.

The oldest of the pyramids, built for Pharaoh Djoser at Saqqara, has a stepped design devised by the architect Imhotep around 2700 BC. Five hundred years later, in the valley of Giza, near modern Cairo, three large pyramids were built for Khufu, Khafre and Menkaure. The characteristics of the Khufu pyramid are as follows:

Shape:	pyramid with square base.
Faces:	isosceles triangles.
Height:	147 m.
Length of base:	230 m.
Slope of faces:	52°.
Slope of edges:	42°.
Orientation:	N–S.

Given the length of the base and the height of the pyramid, we can calculate the slopes of the faces and the edges. However, we can do so thanks to our knowledge of trigonometry, which the Egyptians of the time were not aware of. How did they determine the shape and dimensions of the pyramid?

Let's attempt to resolve three mathematical problems related to the Giza pyramids:

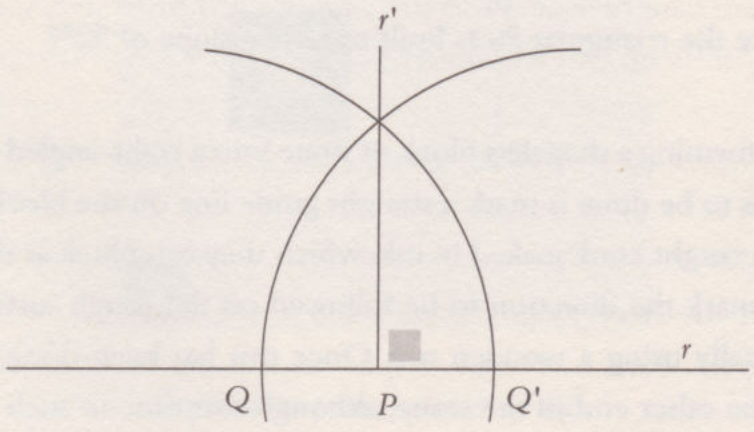
1. How was it possible to sculpt blocks of stone to transform them into right-angled prisms?
2. How was the right angle of the square base of the pyramid marked on the ground?
3. How were the triangular faces built to have a slope of 52°?

When transforming a shapeless block of stone into a right-angled prism, the first thing that needs to be done is mark a straight guide line on the block. To do so, the artisan can tie a taught cord, soaked in ink, which they can pluck as if it were a bow. The cord will mark the direction to be followed on the rough surface, which can be verified visually using a wooden rod. Once this has been done, the operation is repeated at the other end of the stone, although this time in such a way that the

two guidelines that have been marked are parallel, this being determined visually. Having done so, we now have the basis for sculpting the first flat side of the stone. To this day, some builders continue to be guided by the visually determined line as being more accurate than the taught cord.

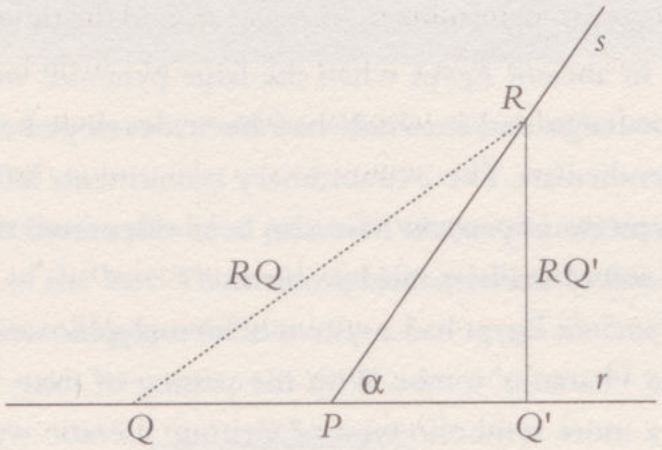
A set square can then be used to complete the work on what will be the other face of the block, and so on, until it is complete. The task is far from simple. Without the right expertise, the block could be reduced to half its original size. We might then ask how the artisan made his or her set square. How could they ensure both sides were perpendicular? This leads to a second problem: How to mark a right angle on the ground. How did the Egyptians construct a right angle more than 5,000 years ago? The triangle with sides of 3 m, 4 m and 5 m is known as the Egyptian triangle. It is believed it was used to determine right angles around the time of the pharaohs. It is still used for this purpose in different parts of the world, including Spain, Argentina and Sweden, albeit in smaller yet proportional versions, such as 30 cm, 40 cm and 50 cm. This could have been the method used to determine the right angles of the corners at the base of the large pyramid.

Another plausible procedure would be to have used the Euclidean method. Euclid lived many years after the pyramids were built – some 2,000 years later – but the method of drawing a perpendicular to a segment may have been known many years before he gave the proof. The same thing occurred with the famous theorem that bears his name. Indeed, some 4,000 years earlier, the Egyptians were able to mark a right-angled corner at the base of a pyramid at point *P*. They would then draw a straight line *r* passing through *P* in the direction of the side. They then took *r* and marked two points *Q* and *Q'* equidistant from *P* (this can be done using a piece of string). And finally, they would use the same piece of string and the same measurement $PQ = PQ'$ (although it could easily be any other) to draw two circular arcs that intersected on the perpendicular line, as shown in the image:



However, some experts in construction, such as Peter Hodges, who have studied the Egyptian methods, believe another procedure was more likely. One reason for this is the fact that in ancient Egypt, the right angle dominated and circles were hardly used. Egyptian constructions remained rectangular until many years later.

From this perspective, right angles would probably have been constructed using the following method. First, we proceed as before, drawing a straight line r passing through point P on which the vertex of the square will be located, marking it at two points Q and Q' equidistant from P. We then mark a point R on a string s, the end of which is attached to P. When the distance RQ is the same as the distance RQ', the string s will be perpendicular to the first line r. In other words, the angle α will be a right angle



This idea is based on the construction of an isosceles triangle whose height is determined by PR.

Finally, how did the Egyptians construct faces with a gradient of 52°? We will not attempt a literal answer to this question. The question is formulated in terms of our contemporary numbers, although the idea is to determine how the faces were given the slope they have. As specialists have noted, the slopes bear closer relation to the height and base than to a specific angle. Given that the tangent of an angle relates precisely to these two lengths,

$$\tan(52°) = 1.2799... \approx 1.28 \approx \frac{147 \text{ m}}{115 \text{ m}} = \frac{\text{pyramid height}}{\text{half of base side}}.$$

does this mean the large pyramid was built in the knowledge the faces would have this gradient? Perhaps the most important thing was that the edges had a slope of 42°.

$$\tan(42°) = 0.90040404... \approx 0.904 \approx \frac{147\text{ m}}{162.6\text{ m}} = \frac{\text{pyramid height}}{\text{half of base diagonal}}.$$

However, if this were the case, why these angles and not others? Are these quantities related to the Egyptian units of measurement at the time? Are they multiples of fingers, palms or cubits? It is hard to say, since the equivalents of these measurements differ depending on the interpretations. For example, it appears the Egyptian imperial cubit, used for the pyramid of Khufu was 52.4 cm, whereas other values of the cubit in subsequent millennia range from 31.6 cm to 51 cm. Accepting the equivalence of the imperial cubit, the large pyramid would have a height of 280 cubits and a base of 440. The proportion between them is 7/11.

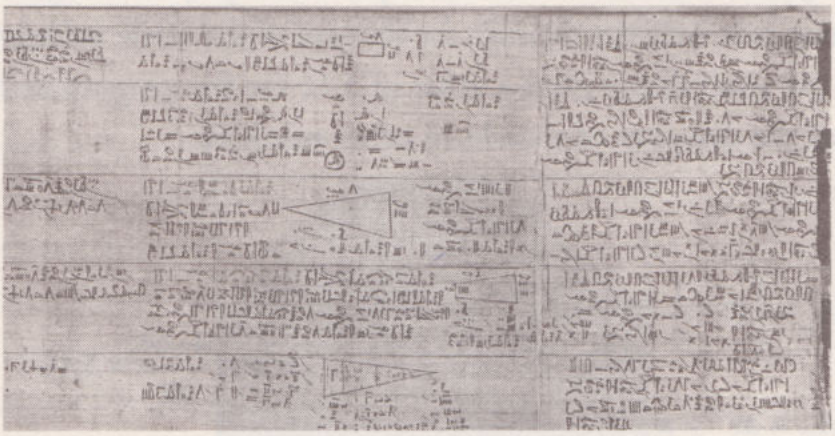
The reason behind this proportion is a mystery. What we can be sure of, however, is that at the time in ancient Egypt when the large pyramids were built, rigorous mathematical knowledge and methods had been developed for drawing lines, parallels and perpendiculars. The extraordinary monuments left behind are solid proof. Fortunately, pieces of papyrus have also been discovered that allow us to be sure the Egyptians solved mathematical problems.

The culture of ancient Egypt had a system of hieroglyphic writing that was used on the walls of the Pharaohs' tombs. With the passing of time, the symbols were transformed into a more symbolic type of writing: hieratic writing. Developed towards the end of the period of the great pyramids, it was used to document a series of aspects from Egyptian culture and life. The works were written on sheets of papyrus. Thanks to these, we know that the Egyptians used a decimal notation and solved problems from geometry and calculation using unit fractions.

Of all the sheets of papyrus that have survived the passing of time, one stands out on account of its important mathematical content. This is the Rhind papyrus, discovered in Thebes in the second half of the 19th century close to the tomb of Ramesses II. It is also known as the Ahmes papyrus, alluding to the name of the scribe who wrote it. He stated that he was copying an older text written by an unknown author or authors. Ahmes' copy has been dated to around 1600 BC, although the original text may date back another three centuries.

The Rhind papyrus contains 87 mathematical problems, the first six of which involve dividing numbers by 10. There are 16 problems dealing with the addition of fractions, 18 for tables and equations, 8 for problems of distribution, 14 for the calculation of the volumes of prisms and truncated pyramids, 5 for the calculation of areas of land and volumes of circular solids, and 15 problems related to economics.

The type of writing is practically identical to that of modern mathematics. Few differences will be found comparing the notes of a mathematics student of any level of education with the Rhind papyrus.



The Rhind papyrus, one of the oldest and best-preserved mathematical texts.

The Egyptians also built cylindrical granaries whose capacity was calculated based on the circular area of the base. Their rule for the calculation was to "subtract one ninth of the diameter and square the result".

Illustration accompanying problem 41 of the Rhind papyrus.

Problem 41 of the Ahmes' papyrus involves calculating the volume of a cylindrical granary 10 cubits high with a diameter of 9 cubits. The result is obtained by multiplying the area of the base by the height. The Egyptian rule is used for this calculation. The ninth part of 9 cubits is 1 cubit. The difference is 8 cubits. Squaring this value gives 64 square cubits, which, when multiplied by 10, gives a total of 640 cubic cubits. However, the exact solution is:

$$V = \pi \cdot \left(\frac{D}{2}\right)^2 \cdot h = \pi \cdot \left(4.5c\right)^2 \cdot 10c \approx 636.1725\ldots\ c^3.$$

The result of the Egyptian approximation is larger by just 0.6%, and is linked to the value of pi implicit in its formula, which is the only factor that differs from the current method of calculation. Some historians value the Egyptian method for this implicit approximation of the number pi. If we compare the formula implicit in the Egyptian rule with the formula for the area of the circle we know today, we can see that the equivalence gives a value of 3.16 for the proportion between the perimeter and the diameter of a circle, in other words pi:

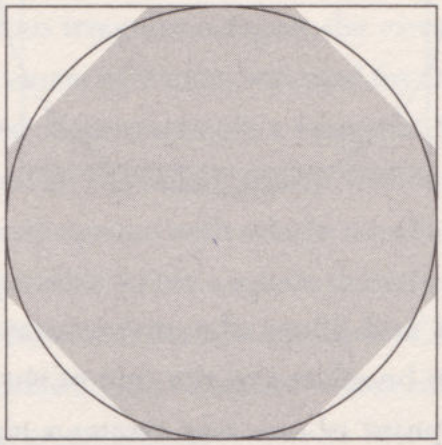
$$\left.\begin{aligned} A_E &= \left(D - \frac{1}{9}\cdot D\right)^2 = \left(\frac{8D}{9}\right)^2 = \frac{64D^2}{81} \\ A_C &= \pi r^2 = \pi\left(\frac{D}{2}\right)^2 = \pi\frac{D^2}{4} \end{aligned}\right\} \Leftrightarrow \pi = \frac{4\cdot 64}{81} \approx 3.16.$$

However, there are two issues that merit attention and are worth valuing over and above a good decimal approximation. On the one hand, the Egyptians interpreted and quantified volumes by multiplying the area of the base by the height; on the other, how did they arrive at this formula? What thoughts left unrecorded by the Egyptian papyrus led them to determine such a formula? One hypothesis relates the Egyptian rule for calculating the area of a circle to the area of an irregular octagon inscribed within a square with sides nine units long.

In the search for a rectangular shape with an area similar to that of the circle, it is clear that the inscribed square is too small, and that the circumscribed square is too large. Taking the arithmetic mean of both does not give a good estimate of the real area of the circle. It is equivalent to taking 3 as the value of pi, π, a value that was used for many centuries in ancient Egypt and Mesopotamia. However, it is enough to just measure one turn of a wheel to see that the relationship between its perimeter and diameter is clearly more than 3.

Bearing in mind that areas, in contrast to lengths, cannot be measured on the ground, another solution would be to draw a circle, measure its perimeter, then calculate the perimeter and compare the results. What formula would be used for calculating the perimeter? Is it reasonable to take the arithmetic mean of the perimeters of the inscribed and circumscribed squares as the perimeter of the circle? It might be. However this leads us to another problem: in the absence of Pythagoras' theorem, it is impossible to calculate the perimeter of the square inscribed in the circle.

One hypothesis is that an irregular octagon was constructed and used as an equivalent to the circle. To do so, the sides of a square of nine units would be divided into three equal parts. The eight thirds would then be joined to give an irregular polygon with eight sides, whose area is visually indistinguishable from that of the circle:



The area of the circle is 63.6 u^2. The area of the irregular octagon differs by less than 1%:

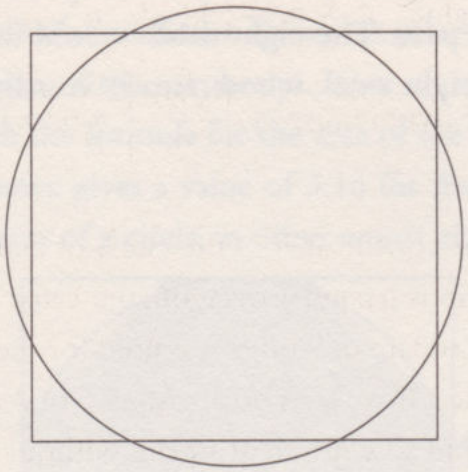
$$A_8 = 9^2 - 4 \cdot \frac{1}{2} \cdot 3^2 = 81 - 18 = 63\ u^2.$$

Another hypotheses is related to problem 50 of the Ahmes papyrus. Here it is assumed that the area of a circular field with a diameter of 9 units is the same as that of another square whose sides are 8 units; we are then referred to problem 48 for the justification. Problem 48 contains a diagram in which an irregular polygon appears inscribed in a square. The number 8 is written in the centre of both shapes, although the design is extremely imprecise. The inscribed polygon has seven, not eight, sides! Furthermore, one of the sides of the polygon does not coincide exactly with another of the square. Regardless, why did the Egyptians believe a circle of diameter 9 was equivalent to a square of sides 8.

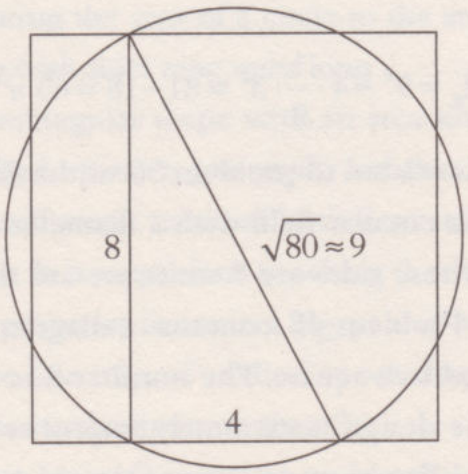
From a modern perspective, both areas are similar:

$$A_{\bigcirc} = \pi \cdot 4.5^2 = 63.617...\ u^2$$
$$A_{\square} = 8^2 = 64\ u^2.$$

We can appreciate the similarity visually:



According to Robins and Shute, the answer may lie in the fact that the diameter of the circle can be related to the side of the square. Joining a vertex of the square to the midpoint of one side creates a right-angled triangle with a hypotenuse that is $\sqrt{80}$. The value is extremely close to $\sqrt{81} = 9$, which is the diameter of the circle:



The curious thing about this last estimate is that taking 9 as the hypotenuse of a right-angled triangle with catheti (short sides) of 8 and 4 gives an even more precise result for the area of the circle than the correct value $\sqrt{80}$, since 64 is closer to 63.617 than 62.83:

Incorrect hypotenuse: $8^2 = 64\ u^2$.

Exact value: $\pi \cdot 4.5^2 = 63.617...\ u^2$.

Correct hypotenuse: $\pi \cdot \left(\frac{\sqrt{80}}{2}\right)^2 = 62.8318...\ u^2$.

Regardless, using 64 u^2 for the area of the circle of diameter $9u$ instead of 63 u^2, as in the case of the previous irregular octagon, the error is smaller.

Is it not strange that a square of 9 units was used for the procedure? Why not use a square of 3, 6 or 12 units? If we start with a square of 3 units, the octagon is 7 u^2, and it is not possible to find an equivalent square without using irrational numbers. Even the areas of the closest squares with whole number sides, such as 4 and 9 are too far off. However, it is possible to use a square the sides of which are a multiple of 3 for the division. However, what is the best multiple of 3? The proportion between the area of the inscribed circle ($A_\circ$) of a square whose sides are of length $3x$ and the corresponding irregular octagon (A_8) is:

$$\frac{A_\circ}{A_8} = \frac{\frac{\pi x^2}{4}}{7x^2} = \frac{9\pi}{28} \approx 1.01.$$

They are extremely similar. Finding a square with an area similar to the octagon means finding a number c such that $c^2 = 7x^2$. This is impossible when c is a whole number, but perhaps it might be possible to find an approximation, $c \approx x\sqrt{7}$, such as $c = 8$. This is the value used by the Egyptians and gives very similar results: $7x^2 = 63$ and $c^2 = 64$.

Rey Pastor and Babini believe the Egyptian rule is based on their skill at making calculations using unit fractions. If the rule involves subtracting the ninth part of the diameter, we must ask what is the whole number fraction of the diameter of the form $1/n$, when n is a natural number, such that it gives the side of the equivalent square. Imagine the diameter of the circle is $D = 1$. Subtracting the fraction $1/n$ and calculating the value of n such that squaring the result gives the area of the circle of diameter 1, we can see that the result is close to 9:

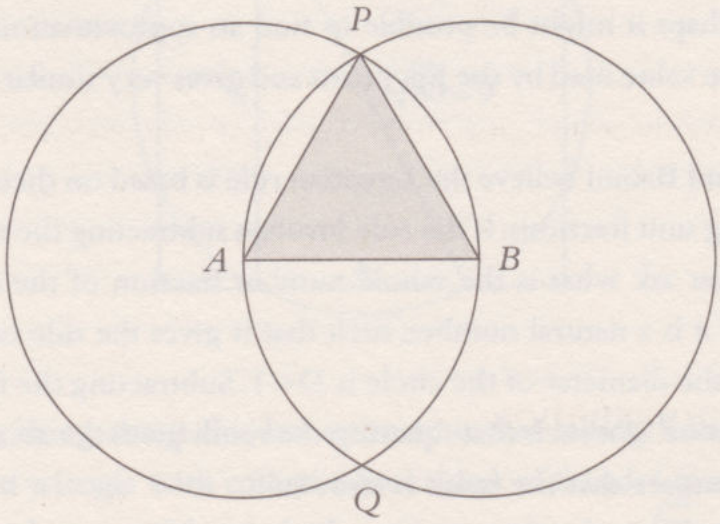
$$\pi \cdot \left(\frac{1}{2}\right)^2 = \left(1 - \frac{1}{n}\right)^2 \Leftrightarrow \sqrt{\pi} \cdot \frac{1}{2} = 1 - \frac{1}{n} \Leftrightarrow n = \frac{2}{2 - \sqrt{\pi}} \approx 8.789...$$

Mathematics with a capital M

A large part of mathematics as we know it today is directly descended from the path established in Euclid's *Elements*. The work does not only record problems and their solutions, but also shows a mathematical way of thinking that was paradigmatic until the first half of the 20th century, when Bertrand Russell shook its foundations. An example of the philosophical and logical nature of the book can be found in proof by reductio ad absurdum.

The criticism of the *Elements* attacks its first line, the definition that the point is that which does not have parts. A point is now defined as an element of an affine or topological space. However, let us consider a criticism of the first proposition, which describes the construction of an equilateral triangle and is often used as an example of the Euclidean method, i.e. first state the theorem and then prove it using the application of established axioms. This is the procedure that could have been used by the Egyptians to mark the perpendicular corners of their pyramids on the ground.

Proposition 1 explains how to construct an equilateral triangle based on a single line segment. We begin with a segment AB. We then use a compass to draw a circle of radius AB whose centre is A. The operation is then repeated with B as the centre, such that the two circles create the points of intersection P and Q, each of which is located at the same distance from A and B. The triangles of the vertices ABP and ABQ are thus equilateral:



The modern criticism of this proof points to the fact that it accepts an axiom of the continuity of lines that does not form part of the Euclidean postulates, and hence it cannot be guaranteed that the two circles will intersect at some point.

Hence, the *Elements* is not the *definitive mathematical work* but a cultural product that brings together and embodies the knowledge of an era whose origins can be found in various cultures. There are even those who dare to claim that it has taught us to think mathematically. However, mathematical thought is not just limited to the triad of axiom, theorem and proof.

Indeed, there are other forms of mathematical thought. In spite of the fact that the *Elements* contains an algorithmic procedure, such as the procedure for finding the greatest common divisor of two natural numbers, it is impossible to claim that algorithmic thought forms a true part of the mathematical thought of this work. In the book on algebra, there are no iterative processes that converge on the solution to a problem. These ideas come later on and are characteristic of the cultures of China, Arabia and India. Eudoxus, who may have been one of Euclid's contemporaries, worked in this area of thought, but his work does not form part of the *Elements*. Archimedes, who lived a century after Euclid was arguably the first to use the idea of successive convergent approximations to give the most precise calculation of the area of a circle at the time. Prior to Archimedes, Eudoxus had worked in this area, and the idea of succession and controlling its convergence would give rise to infinitesimal calculus some 2,000 years later. The question arises of whether Euclid would have regarded infinitesimal calculus as a mathematical process or idea.

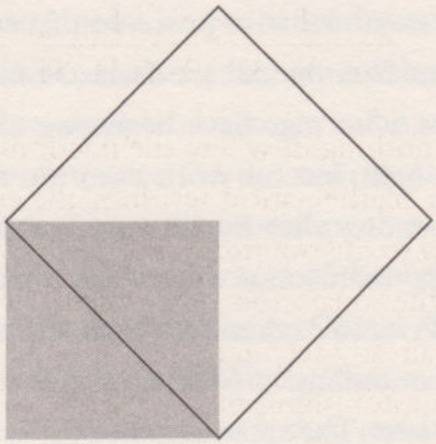
Bertrand Russell went further, to the point of saying that mathematics can be deduced from logic. However, the fact that it can be deduced from logic does not mean that logic is its essence. Every day, we make decisions that can be justified using logic, but that are not made based on logic. We make our decisions for many reasons, including logic. However, many of our decisions are made based on experience, intuition, imitation, advice and an endless number of reasons that can be validated by rational thought after the fact. However, this is not the only way we think. Similarly, nor can mathematical thought or the development of mathematics can be reduced to logic.

Successive approximations

The *Shulba Sutras* are the only Indian mathematical documents from the Vedic period – between the 2nd and 8th centuries BC. They include instructions for the precise construction of square and circular altars for worship in the home. However, in public life, the altars had to be more sophisticated, incorporating triangles, rhomboids and trapezoids. One of these combines these basic polygons to create a

figure in the shape of a bird, perhaps in the hope that a sacrifice offered on such an altar would cause the soul to fly to the heavens.

One problem was the construction of altars the area of which was double another given area. This is a simple geometric problem that can be solved in two different ways. One is visual, the other numeric. The latter is important when it comes to determining the quantity of material required for its construction. The visual solution is immediate. It is enough to draw a square on the diagonal of the first that will contain exactly four halves of the previous square:



The numeric solution involves the application of Pythagoras' theorem or finding a number that is the square root of 2. In other words, for what length of side x does the square double the area of the other side c? Let us see:

$$\left.\begin{array}{l}\text{Area squared } c\text{: } c^2\\ \text{Area squared } x\text{: } x^2\end{array}\right\} \Rightarrow x^2 = 2c^2 \Rightarrow x = c\cdot\sqrt{2}$$

The *Shulba Sutras* contain instructions for an algorithmic procedure to calculate the square root of 2 using successive approximations. The procedure involves adding a third of the side to its measurement; then adding a quarter of the third and, finally, subtracting the 34th part of a quarter of a third. In other words, if c is the measurement of the side of the square we wish to double:

$$c+\frac{1}{3}\cdot c+\frac{1}{4}\cdot\left(\frac{1}{3}\cdot c\right)+\frac{1}{34}\cdot\left(\frac{1}{3}\cdot\frac{1}{4}\cdot c\right).$$

Summarising this operation, we can see that the result is an extraordinary approximation to the square root of 2, since it coincides with the correct value for the first five decimal places.

$$1+\frac{1}{3}+\frac{1}{12}-\frac{1}{408}=1.41421568...$$

Subsequently, in the 15th century, two more terms were added to this approximation to make it correct to seven decimal places:

$$\frac{-1}{3\cdot 4\cdot 34\cdot 33}+\frac{1}{3\cdot 4\cdot 34\cdot 34}$$

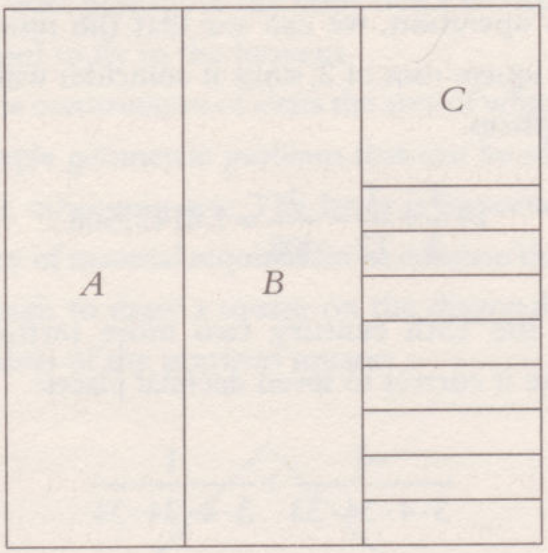
The *Shulba Sutras* says nothing of where the fractions and the number 34 come from. As in so many other mathematical writings, the results are documented but the creative process leading to the solution is not. One hypothesis claims this Indian algorithm for calculating the square root of 2 was based on the procedure used by the Babylonians. We have already seen how the latter had managed to calculate the diagonal of a square with astonishing precision, but we have no evidence of the method they used. We do not even know if the ideas were algebraic or geometric.

Just how do mathematicians devise a theory of the creative process by which a problem was solved? We find ourselves obliged to imagine a hypothetical path, the origin of which is the endpoint of the journey of the person who solved the problem. To know the thoughts of the author of the solution written in the *Shulba Sutras* means giving a meaning to the fractions and the numbers involved.

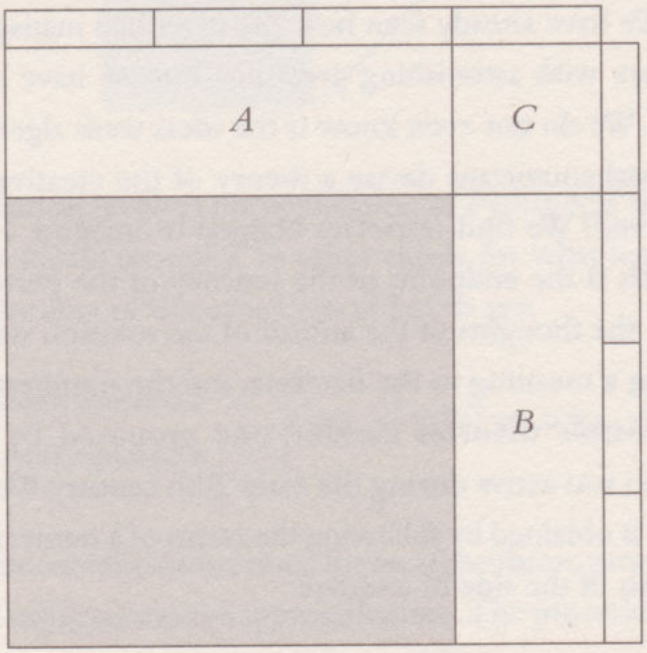
The most plausible theories include one proposed by Datta, an Indian mathematician who was active during the early 20th century. We begin by thinking the approximation is obtained by following the terms of a numerical series that starts with the unit length of the side of a square:

$$\{1,\ 1.33333,\ 1.41467,\ 1.4142157,\ 1.4142135\} \to \sqrt{2}.$$

For a square of length one, the area is also one unit. Given that the first step consists of adding a third, we divide the square into three equal parts, giving three rectangular strips. Let A and B be the first two strips and we divide the third into three equal parts. Each of these will be a square. Let's call the top one C and divide each of the bottom squares into four parts. We now have the following configuration:



We now have eleven strips (A, B, C and the eight smaller ones) that we arrange around the original square as follows:



Completing the missing corner gives a square the area of which will exceed the duplication we are after by precisely the area of this corner, since the strips that have been added give the area of the original square. However, we can see that if we add the small corner, the length of the resulting square is exactly that of the one given in the *Shulba Sutras*:

$$1+\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{4}.$$

Datta explains the use of the fraction $1/(3\cdot 4\cdot 34)$ from a more Western algebraic perspective. According to Datta, the explanation lies in the fact that the missing corner is a remainder distributed among the two sides from which it is derived. This means the area of the corner, which is $1/12^2$, is split across two rectangles and a new corner of length x, which will be removed from the upper and right sides of the piece:

$$2\cdot x\cdot\left(1+\frac{1}{3}+\frac{1}{12}\right)-x^2=\left(\frac{1}{12}\right)^2.$$

Having reached this equality, Datta's argument is that the area of the new corner that will be left empty (this small square of side x) can be discounted as it is so small, hence:

$$2\cdot x\cdot\left(1+\frac{1}{3}+\frac{1}{12}\right)=\left(\frac{1}{12}\right)^2\Rightarrow x=\frac{1}{12\cdot 34}.$$

Perhaps these were the thoughts of the Indian author, but the algebraic argument and the logic of discarding extremely small quantities are hard to reconcile with the search for increasingly precise values. Getting under the skin of the Indian author or, better put, into their mind, means searching for the geometric ratio of the strange factor of the denominator, which we know already to be 34. The problem involves dividing the small square corner whose side is 1/12 into enough parts so that it fits into the upper and right sides of the piece, which are 16 + 16 = 32 parts. Removing $1/(12\cdot 32)$ from each of the 16 small squares that make up the profile of the piece gives another polygon inscribed in a square whose side will be:

$$1+\frac{1}{3}+\frac{1}{12}-\frac{1}{12\cdot 32}.$$

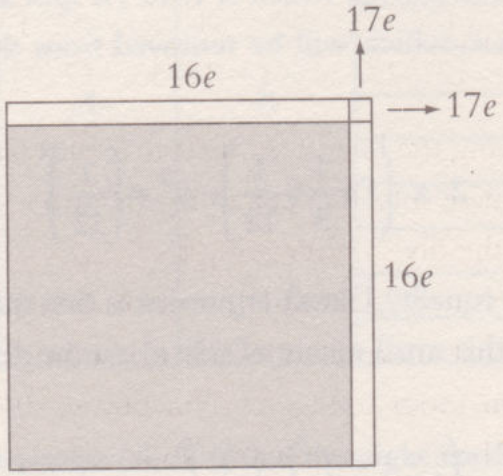
The area of the square will be extremely close to the desired value:

$$\left(1+\frac{1}{3}+\frac{1}{12}\right)^2=2.00694...\Rightarrow\varepsilon=+0.35\%$$

$$\left(1+\frac{1}{3}+\frac{1}{12}-\frac{1}{12\cdot 32}\right)^2=1.99957...\Rightarrow\varepsilon=-0.022\%.$$

The fact that the value of 34 is still missing is perhaps due to the fact that attempts were made to improve the calculation, although there is another more like-

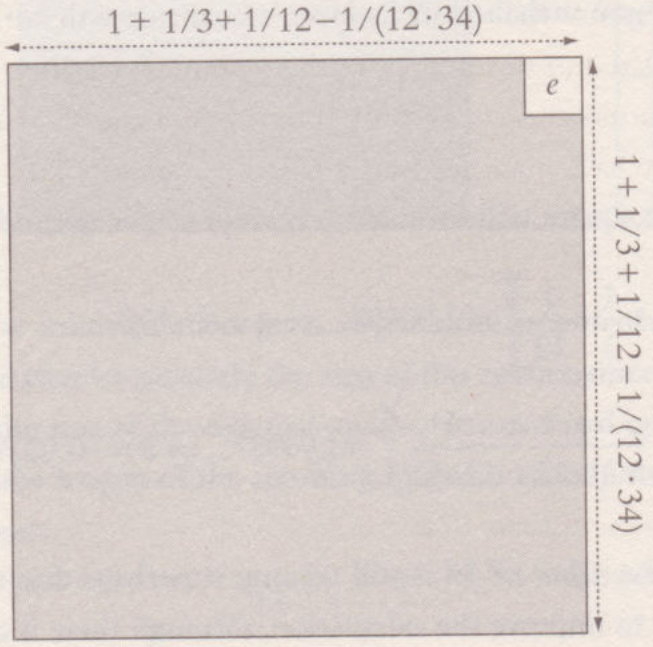
ly possibility. Instead of reducing the sides of the irregular polygon above, we cut the square of $1+1/3+1/12$ on its top and right sides. The small corner fits precisely 17 times into each of these:



Hence, we have divided it into 34 strips to remove 17 from the top side and another 17 from the right side of the large square. By doing so, we will have removed a surplus in the form of a tiny square corner of side:

$$\frac{1}{12\cdot 34}.$$

The resulting shape is once again an irregular polygon inscribed in a square, the sides of which are precisely the approximation given in the *Shulba Sutras*:



The play between the surplus of removing 34 corners and the shortfall of removing 33 appears to give rise to the alternation between the numbers 33 and 34 that characterises the following approximations of the Indian procedure:

$$1+\frac{1}{3}+\frac{1}{12}-\frac{1}{12\cdot 34}-\frac{1}{12\cdot 34\cdot 33}+\frac{1}{12\cdot 34\cdot 34}.$$

However, and following the line of thought that runs parallel to the Indian one, dividing the original square into five parts provides a better initial approximation to the duplication than the division into thirds.

This line of thought is not Euclidean. In spite of being logical and deductive, it is not based on axioms that are applied to arrive at a previously determined result. Here there is no theorem, proof and conclusion, but we discover the true nature of something as our search brings us ever closer to it.

Ethnomathematics: mathematics as a cultural phenomenon

Mathematical thought is most complex and profound in cultures with a written language and it is directly associated with this capability. For cultures for which we have written documents, we know how they thought, although not exactly, since their mathematical writings do not reflect everything. In fact it appears that what is missing is precisely what Euclid wrote in his *Elements*. In other words, the acts of thought that relate cause and effect.

In the Egyptian pyramids, we see the square and not the circle. In Stonehenge, we see the circle and not the square. Is it perhaps the case that the square is a shape for monuments related to death, such as the pyramids? Might the circle be related to astronomical matters and rituals related to the Sun and the Moon?

The cultures we have discussed in this first chapter ceased to be a very long time ago. Their mathematical ideas were developed long before the advent of what we now call Western culture. Their development was a local phenomenon: each culture developed its own mathematics and solved the problems they were faced with autonomously and indigenously.

Our idea of what mathematics is and how it has arisen is closely related to the idea of a highly continuous development in space and time. However, it appears this was not the case in prehistoric times.

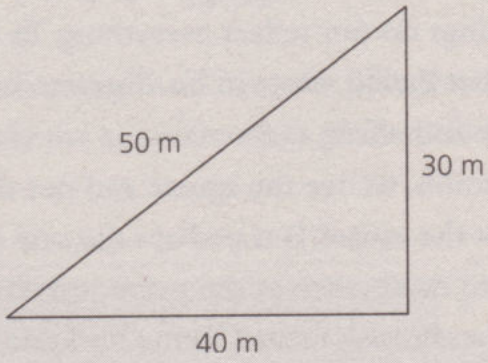
This is how everything begun in our culture. What exists and what has existed beyond it? Before Columbus discovered America, there were cultures there that had

developed important mathematical knowledge. And after the discovery of the new continent, to this day there remain cultures that are different from the West that have also developed mathematical knowledge that has also survived. Let us now turn our attention to these ideas.

RURAL MATHEMATICS

Towards the end of the 1980s, Professor Guida de Abreu studied the mathematical procedures used by peasants in the north-east of Brazil. The discrepancies between academic and indigenous real-word methods represented an obstacle for the implementation of new agricultural technology in the region. The area of a triangle, for example, was calculated by multiplying the average of the length of its two sides by half of the other. In other words, $(x+y)\cdot z/4$.

This method involves some risks. In the case of an equilateral triangle whose sides are x, the area obtained using this method would be $S=x^2/2$, which differs from the real value of $(x^2\sqrt{3})/4$. For the triangle with sides of 30 m and 40 m and a hypotenuse of 50 m, the three possible options give three different areas. The real value is 600 m^2. The values using the rural procedure are $S_1=800$ m^2, $S_2=875$ m^2 and $S_3=675$ m^2.



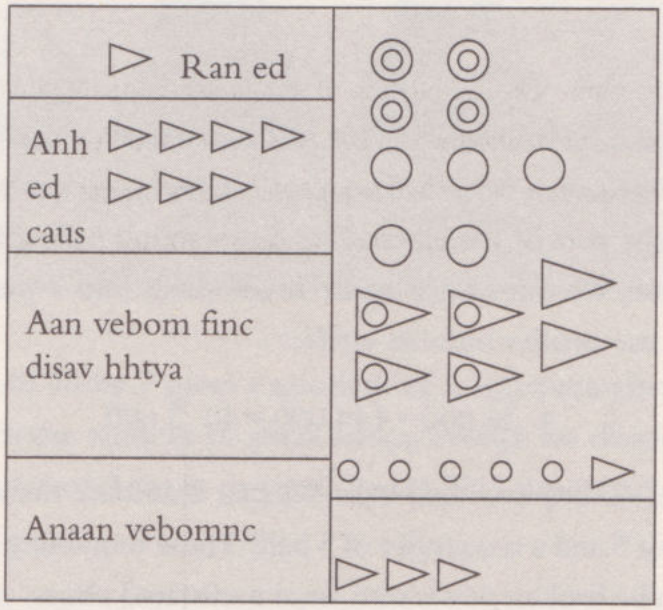
Given that the final value is closest to the correct one and has been obtained by taking the average of the longest two sides, we are lead to think that this technique would give more accurate results. It is certainly much more practical than trigonometric calculation. Furthermore, the system of units of measurement used in this agricultural context was based on the *braça*, the *cubo* and the *conta*. A braça could vary between 2 m and 2.20 m, and corresponded to the standard measurement taken using a wooden stick. The cubo was the area of a square with sides of one braça. The conta was the surface of a square with sides of 10 braças.

Chapter 2

Counting and Calculating More and Better

Written numbers and calculations

What would we think if we came across the following piece of paper on the ground while walking along the street?



It is a reproduction of a Sumerian tablet dating back more than 4,600 years and discovered in Shuruppak in Iraq. According to Georges Ifrah, it may be the oldest written example of division. This mathematician and historian has written large and meticulous studies of numbering and calculating systems throughout the world, systems that were developed long before mathematics was given its name.

The tablet deals with the distribution of barley among a number of men. The left column contains the quantity of barley to be shared out: one granary and seven *silos* (one granary was equivalent to 1,152,000 silos). The right column contains the calculations required to determine the distribution. The interpretation of the text of the tablet is that after sharing out one granary of barley among the various men, they are entitled to 7 silos each. This means there are 164,571 men and 3 silos are left over.

The original tablet uses hand-drawn geometric shapes to represent the numbers in the division. Balls and cones, some small and others larger, are used to represent numbers. The small cone corresponds to one unit; a ball corresponds to 10 units; a large cone corresponds to 60 units; a perforated large cone corresponds to 600; a large ball to 3,600; and a perforated large ball to 36,000 units.

The division process on the tablet is carried out as follows. The numerator is 1,152,000. Broken down into powers of 60, it can be written as

$$1{,}152{,}000 = 5 \cdot 60^3 + 2 \cdot 10 \cdot 60^2.$$

However, instead of being expressed in this way, and since there are no known objects for representing larger quantities, the largest unit at the time, in other words 36,000, was used. We need 32 perforated balls to represent 1,152,000:

$$1{,}152{,}000 = 32 \cdot 36{,}000.$$

If we distribute the 32 balls into 7 equal parts, we can see we need 4 and another 4 are left over. The 4 for each man are the quotient and correspond to the perforated balls in the top right part of the tablet. The 4 remaining balls are the remainder of this first distribution. We must once again divide them into 7 parts. Since there are $4 \cdot 36{,}000$, we can use smaller units to write:

$$4 \cdot 36{,}000 = 144{,}000 = 40 \cdot 3{,}600.$$

In other words, 40 unperforated balls. We can distribute them into groups of 7, giving a quotient of 5 and a remainder of 5 balls. These remaining balls, equivalent to $5 \cdot 3{,}600$ units, can be broken down into large perforated cones, each corresponding to 600 units:

$$5 \cdot 3{,}600 = 18{,}000 = 30 \cdot 600.$$

We have 30 large perforated cones to be distributed among 7 people. The quotient is 4 and the remainder 2. This means 2 large perforated cones of $2 \cdot 600 = 1{,}200$ units are left over, which must once more be distributed among 7. To do so, we shall use the stone immediately below, which is the unperforated cone, corresponding to 60 units:

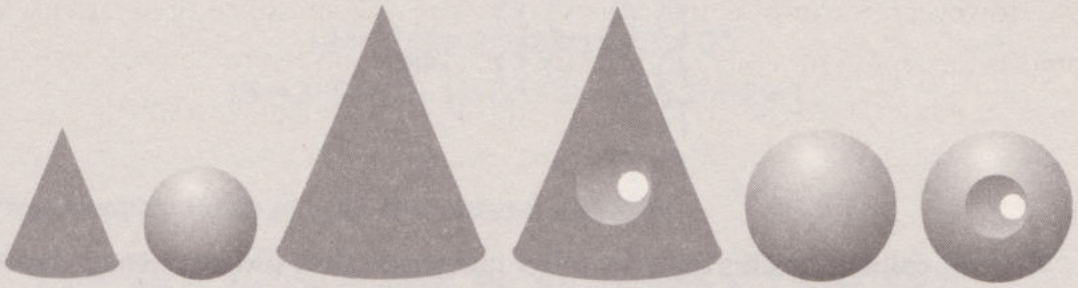
$$1{,}200 = 20 \cdot 60.$$

These 20 cones are divided among 7, which gives a quotient of 2 remainder 6, meaning there are $6 \cdot 60 = 360$ units left over. This is equivalent to 36 balls, each corresponding to 10 units:

$$360 = 36 \cdot 10.$$

SUMERIAN CALCULATING STONES

The calculating stones (calculii), or pebbles, were small objects of stone or clay, the shapes and sizes of which were used to represent quantities. Their use gave us the word for the act of 'calculating'. Each of these pieces were calculating stones. The Sumerians used small cones and balls as calculating stones, which were also perforated to distinguish a numeric value. What are now referred to as 'renal calculii', or kidney stones, are small concentrations of solid mineral in the kidney. Generally speaking, kidney stones are calcifications and the inconvenience they cause is directly related to their size.



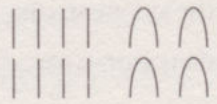
The division of 36 by 7 gives a quotient of 5 and a remainder of one ball with 10 units, which is the same as 10 small cones. Finally, we divide these by 7, giving the final remainder of 3 units or small cones. The process is summarised in the following table:

Stones (calculii)	Quantity	Units corresponding to each person	Leftover units (remainder)
Perforated balls	4·7	4	
Balls	5·7	5	
Perforated cones	4·7	4	
Cones	2·7	2	
Small balls	5·7	5	
Small cones	1·7+3	1	3

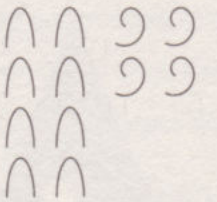
The entries in the top cell of the right column of the tablet correspond to the third column of the table. Below these pieces are the 3 small cones corresponding

to the remainder of the division (fourth column of the table). Hence, this is a comprehensive example of division.

By around 2000 BC, multiplying or dividing by 10 was easy for Egyptians. All they needed to do was substitute each symbol of the digits of the number in question for the corresponding larger or smaller symbols. Consider, for example, the expressions for 48 and 480 in the following image (the Egyptians wrote from right to left):



48



480

When it came to multiplying by other quantities, they did not use an algorithm like ours, but followed a procedure based on doubling or multiplication by 2. Two columns were used to take the product of 117 and 14. The left column contained the successive powers of 2 and the right column contained the doubles of 14. The columns stopped just before the powers of 2 that exceed the quantity by which the number 14 was being multiplied, in other words 117:

1	14
2	28
4	56
8	112
16	224
32	448
64	896

The next step involves finding a combination that adds up to 117 in the left column:

$$1 + 4 + 16 + 32 + 64 = 117.$$

This means that the result of the multiplication is the sum of the numbers in the right column that correspond to these values:

$$14 + 56 + 224 + 448 + 896 = 1{,}638.$$

What we are doing in the left column is looking for the base 2 expression of the largest factor of the product:

$$117 = 1\cdot 2^6 + 1\cdot 2^5 + 1\cdot 2^4 + 0\cdot 2^3 + 1\cdot 2^2 + 0\cdot 2^1 + 1\cdot 2^0 = 1110101_{\text{(base 2)}}.$$

This expression gives us the result. Some 4,000 years ago, the Egyptians, albeit unconsciously, used a change of base in order to multiply. The success of their procedure is based on the fact that it is always possible to find a series of values in the left column that add up to the required number, or in other words it is always possible to express a natural number in base 2. We can see the cause of this experimentally

$$12 = 2^2\cdot 3 = 2^2\cdot(2+1) = 2^3 + 2^2.$$
$$15 = 3\cdot 5 = (2+1)\cdot(2^2+1) = 2^3 + 2^2 + 2 + 1.$$

This property holds for the first natural numbers:

$$1 = 2^0,\ 2 = 2^1,\ 3 = 2^1 + 2^0,\ 4 = 2^2,\ 5 = 2^2 + 1,\ 6 = 2^2 + 2^1,\ 7 = 2^2 + 2^1 + 2^0, \ldots$$

If n is a natural number for which the property holds, it will also hold for the following number, $n+1$. In fact, if n is even, the terms of which it is composed do not include $2^0 = 1$. Hence, this is the power of 2 that must be added to give the next number $n+1$. In this way, $n+1$ will be a sum of powers of 2. In the second case, where n is odd, its decomposition into powers of 2 will end with 2^0. The next number $n+1$ will be formed by adding one unit, in other words, another 2^0 term. Taken together with the existing term gives $2^0 + 2^0 = 1 + 1 = 2 = 2^1$. If the term 2^1 already appears in the breakdown, it is added again, giving another term of 2^2. In all cases, the result will be an addition of powers of 2.

The pattern is clear if we write the powers with which the first 10 natural numbers are written in base 2:

Power of 2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0		*		*		*		*		*		*		*		*	
1			*	*			*	*			*	*			*	*	
2					*	*	*	*					*	*	*	*	
3									*	*	*	*	*	*	*	*	
4																	*

At that time the Egyptians also divided in a similar way, applying the reverse process, by considering division as if it were multiplication. For example, in order to divide 92 by 9, we must find the number by which 9 should be multiplied to give 92. First of all, we write the corresponding columns. On the left, we have the successive powers of 2; on the right we have the doubles of 9 all the way up to 92:

1	9
2	18
4	36
8	72

We now search for an addition that gives 92 in the right column. As no such addition exists, the division is not exact. The closest addition is $18 + 72 = 90$. Hence, the quotient of the division is $2 + 8 = 10$ (the sum of the powers of 2 corresponding to the values 18 and 72), and the remainder is 2.

Counting in and with other places

To count, we first need to give quantities some names. All languages have them, although there are those that lack, or lacked, a specific form of writing to represent them. Today, one set of number symbols are almost universal and used throughout the world. The terms people use to count and designate numbers are also the same. However, the fidelity of a translation may not correspond to fidelity to the concept.

A few centuries ago, many Europeans believed Africans were scarcely able to count to more than 10. This myth was debunked by reports from traders in the 18th century and anthropological studies undertaken during the 20th century.

Some might believe the Kpelle people of central Liberia and Guinea lack numeric capabilities, since they use piles of pebbles for arithmetic operations. However, in a study carried out by Gay and Cole, the Kpelle obtained better results when it came to estimating the number of pebbles in various sizes of piles than American students from Yale.

We use a decimal numbering system for counting and calculation, which we express both orally and in writing. In our society, it is frowned upon for adults to use their fingers to count, something that is only tolerated during the learning period of children.

NUMERIC SIGN LANGUAGE IN AFRICA

The Zulus form the largest ethnic group in southern Africa. They are mostly found in South Africa, although there are also groups of Zulus in Zimbabwe, Zambia and Mozambique. Kamba is a language from the Bantu family that is spoken in Kenya and Tanzania in East Africa. The following table shows the gestures used by both peoples to indicate the numbers from 1 to 10.

Number	Zulu (southern Africa)	Kamba (Kenya)
1	Holding out the left little finger	Holding out the right index finger
2	Holding out the left little finger and middle finger	Holding out the right index and middle fingers
3	Holding out the three outer fingers of one hand	Holding out the right index, middle and ring fingers
4	Holding out four fingers	Separating the index-middle and ring-little finger pairs to form a 'V' with the right hand
5	Holding out five fingers	Right fist closed
6	Holding out the right thumb	Taking the the left little finger with the right hand
7	Holding out the right thumb and index fingers	Taking the left little finger and ring finger with the right hand
8	Holding out three fingers of the right hand	Taking the left little finger, ring and middle fingers with the right hand
9	Holding out four fingers of the right hand	Taking the four fingers of the left hand with the right hand
10	Holding out all fingers	Both fists closed

We give written and verbal expression to our numbers using symbols and phonemes, which reveal the decimal base of our system. The numbers 1 to 10 all have different representations. After 10, the phonetic roots determine the verbal expression of each quantity. For example, from 11 to 19:

11	12	13	14	15	16	17	18	19
eleven	twelve	thirteen	fourteen	fifteen	sixteen	seventeen	eighteen	nineteen
1-ce	2-ce	3-ce	4-ce	5-ce	10+6	10+7	10+8	10+9

The same thing happens for successive powers of 10, the base of the system in which the first syllables indicate the number: thirty (30), forty (40), eighty (80), two hundred (200), three hundred (300), four thousand (4,000), one hundred thousand (100,000). An expression such as 'seven thousand three hundred and fifty two' implies the decimal breakdown of the quantity to which we are referring:

$$7 \cdot 1{,}000 + 3 \cdot 100 + 5 \cdot 10 + 2.$$

There are differences within our Western culture. In spite of using the same decimal numbering system, in French the spoken form for tens over fifty is based on 20. Eighty is 'quatre-vingt' or, in other words, four times twenty. Eighty-five is 'quatre-vingt cinq', or, $4 \cdot 20 + 5$.

Claudia Zalavsky was a pioneer in the study of vernacular mathematics, contributing to the early development of ethnomathematics. His book *Africa Counts* is a study that anticipates many of the aspects that would form what Ubiratan D'Ambrosio would call ethnomathematics years later. Zalavsky collected many mathematical ideas that had been born in the heart of African cultures – systems of non-decimal numbering, digital calculation and geometric patterns, for the purposes of both construction and ornamentation.

Beyond our Western culture, oral expressions that use the number five as a basis for representing quantities greater than those that can be counted using our fingers are commonplace. In certain linguistic varieties of Bantu (originating in Central Africa), the number 5 is *tano* and determines the terms for 6, 7, 8 and 9. These are constituted by adding the endings 1 *(-mwe)*, 2 *(-vali)*, 3 *(-tatu)* and 4 *(-ne)* to a prefix of 5 units. This gives the terms for 6 *(tano-na-mwe)*, 7 *(tano-na-vali)*, 8 *(tano-na-tatu)* and 9 *(tano-na-ne)*.

In Guinea-Bissau and elsewhere in Central Africa, systems based on five and twenty are also used, with five being understood to correspond to the fingers of a hand, and twenty understood to be the total of a person's hands and toes. This allows us use the term 'two hands' to refer to 10 or a 'full man' to refer to 20. An expression such as 'five full men' would mean 100.

The way in which a culture refers to numbers denotes the way in which it thinks. These indigenous terms are practical for counting small quantities, but not when it comes to carrying out operations involving large numbers. The numbering system of the Igbo people (Nigeria) is based on twenty. The term for the square of 20 (400) is *nnu*, and the square of 400 is expressed as *nnu khuru nnu*, which means '400 meets 400'.

One context in which numbers play a fundamental role is that of commerce. In order to trade, it is necessary to know how to measure and weigh, calculate, and have a system for records. Trade is not possible without exchange, which gives rise to the need for a unit of value. This leads us to multiplication and division. In Africa, shells, cows, salt, slaves and gold have all been used as a currency. At present, money prevails, although various objects are also directly exchanged in local markets.

One century ago, the Ewe on the coast of Guinea used cowry shells for trading. A *hoka*, the Ewe unit of exchange, comprised 40 shells. However, further inland, the hoka corresponded to 35 shells, not 40. The Ewe were well versed in multiplication and were able to carry out operations quickly, taking 20 times 3 shells and adding 10 to give 2 inland hokas: $20 \cdot 3 + 10 = 70$.

Does this mean the Ewe created the relationship between both hokas? Given 20 is half of 40 and 10 is a quarter, did they know that three halves and a quarter of their hoka gave double the other hoka? Furthermore, were they aware that the ratio between both currencies was 8:7 and that the inland equivalent of a price in Ewe hokas was given by multiplying this by 7 and dividing it by 8? It is hard to answer this question.

$$3 \cdot \frac{1}{2} h_E + \frac{1}{4} h_E = 2h_I \Leftrightarrow \frac{7}{8} h_E = h_I.$$

The Yoruba numbering system

The extraordinary and complex numbering system of the Yoruba people of Nigeria deserves special consideration. By means of example, it suffices to state that its term for 48 literally means, $20 \cdot 3 - 10 - 2$.

The Yoruba use a base 20 system but, in contrast to the vast majority of base 20 systems, numbers are represented based on subtraction rather than addition. This may seem surprising and overly complex, but it is not the only case of subtractive numbering – it is also used in Roman numerals:

Roman numerals		
4	IV	5–1
9	IX	10–1
44	XLIV	(50–10) + (5–1)

How is a number expressed in the Yoruba system? First of all, we must analyse the expressions of all the numbers up to 20, the base of the system. The numbers from 1 to 10 use a different term for each figure. The terms for 11 to 14 are formed by adding the ending *-laa* to the terms for 1 to 4. It is upon reaching the number 15 that subtraction appears for the first time in order to represent terms with a literal meaning that is $20-5$, $20-4$, $20-3$, $20-2$ and $20-1$. The number 20 has a new term; from 21 onwards, the additive notation is used again, changing to subtractive upon reaching 25. The pattern is repeated successively and cyclically. For example, $105 = 6 \cdot 20 - 10 - 5$ and $315 = 400 - 20 \cdot 4 - 5$ (400 has a specific number).

When it comes to expressing large numbers:

Yoruba (large numbers)	
100	$5 \cdot 20$
200	200
300	$20 \cdot (20-5)$
400	400
2,000	$10 \cdot 200$
4,000	$2 \cdot 2{,}000$
20,000	$10 \cdot 2{,}000$
40,000	$2 \cdot 10 \cdot 2{,}000$

The reasons behind this way of thinking may perhaps lie in the counting of Cowry shells. The count began by forming groups of five and then groups of 20. Five groups of 20 gave a group of 100. When we group the shells together to make a group of five, what we are really doing is counting from 1 to 5. This explains why the Yoruba added these units to 10 to give the numbers 11, 12, 13 and 14. However, it does not explain why the change occurs at 15.

One possible explanation is that only one hand was used for counting. Imagine we are thinking of the number 10 and we use the fingers of one hand to count 11, 12, 13 and 14. How can we use the same hand to count the remaining numbers up to 20? First of all, we hold out our fifth finger so that our hand is completely open. We then close our fingers until we reach the next set of 10. Hence, holding out our all our fingers gives the first 10, and closing them gives the second. Hence,

when we hold our fifth finger out, we are already subtracting 5 from 20, or rather, $20-5=15$. Closing one finger gives $20-4=16$; closing another gives $20-3=17$. When we have all our fingers closed, we are now starting to count the following 10, or in other words, 20.

In a market in Mozambique

Various arithmetic studies have analysed how people calculate outside the academic realm. One of these studies aims to discover how women calculated addition and subtraction in their head as part of their everyday activities. The market is one of the female domains in African culture. To subtract 5 units from 62, more than half the women in the market in Mozambique first subtracted 2 units then subtracted 3 from the result.

$$62-5=(62-2)-3=57.$$

With respect to the same operation, approximately one third subtracted 5 from 60 and then added two units to the result:

$$62-5=(60-5)+2=57.$$

A minority subtracted 10 from 62 and then added the difference between 10 and 5 to the result:

$$62-5=(62-10)+(10-5)=57.$$

For multiplication, the majority of the women doubled the numbers until obtaining approximations to the result. For example, to calculate $6\cdot 13$, one solution is as follows (the method bears a certain resemblance to the Egyptian method mentioned at the start of this chapter):

$$\left.\begin{aligned} 2\cdot 13&=26\\ 4\cdot 13&=2\cdot 26=52\end{aligned}\right\}\Rightarrow 6\cdot 13=26+52=78.$$

However, it is unknown if these calculating procedures were devised by the women themselves, or were adapted from other methods that were previously known, or if they formed part of a cultural or commercial tradition related to their work at the market. Nor is it known how the teaching and learning process occurred, if such a process existed.

In Nigeria, cases of informal calculation similar to those set out above also occur. Some methods for calculating 18 + 19 are:

$$18+19=(18-1)+(19+1)=17+20=37$$
$$18+19=(20-2)+(20-1)=20+20-(2+1)=40-3=37.$$

To divide 45 by 3, knowing that 21/3=7 can be extremely useful:

$$\frac{45}{3}=\frac{21+21+3}{3}=7+7+1=15.$$

These procedures clearly show there are many ways to do things and that mathematical thought exists outside the school.

On an Indian bus

Chennai, formerly known as Madras, is the capital of the state of Tamil Nadu, in south-east India. Bus drivers in the district are required to carry out mental arithmetic with great agility. On the one hand, they need to determine the amount to charge each passenger, which depends on the different prices between different points along the route. On the other, at the end of the day, they must calculate what is known as their *batta*, which is their salary based on the money they have collected during the day. The batta depends on a number of variables, such as the type of bus, the number of journeys and the total value of the takings for the day.

Nirmala Naresh, at the State University of Illinois in the United States, has studied the procedures used by bus drivers to calculate both their batta and the amount that must be paid by each passenger depending on their journey. For this to be possible, they need to be constantly aware of the relationship between the Indian currency, the rupee, and its one hundredth part, the paisa and, above all, the different banknotes and coins:

Notes (Rupees)	Coins	
1, 2, 5, 10, 50, 100, 500, 1,000	Paise	Rupees
	5, 10, 20, 50	1, 2, 5

A street in Chennai, in Tamil Nadu, India (photograph: geekandchick.cl).

Here are some examples of the mental techniques used by bus drivers in Chennai to multiply $3 \cdot 293$ and $3.50 \cdot 61$:

$$3 \cdot 293 = 3 \cdot 300 - (3 \cdot 7) = 900 - 21 = 879.$$

$$3.50 \cdot 61 = 3 \cdot 61 + \frac{1}{2} \cdot 61 = 183 + 30.50 = 213.5.$$

As we can see, the multiplications are not direct and do not use academic methods, but are instead based on the reduction to simpler products that can be easily calculated in their head. The first example seeks a round number close to 293, in this case 300. This can be easily multiplied by 3 in our head, but after doing so, the result exceeds the correct answer by exactly 3 three times the excess using 300 instead of 293 – in other words, 7 units. Hence the need to subtract three times 7 from 900. In the second example, the decimal number 3.50 is broken down into its whole number and decimal components, or rather three units and one half. We then proceed to multiply 61 by 3, which gives 183. Finally, it is necessary to add half of 61, which gives 30.5.

This mental arithmetic represents a great skill, not just in terms of additive decomposition, but also in terms of the real and practical use of what is known in academic speak as the distributive property of the product with respect to the sum. In spite of the fact that these people have received an extremely basic mathematical

education, in which they would have studied mental arithmetic at school, the procedures used at work and in their everyday life do not correspond to the academic ones. As such, they are vernacular.

Separating the whole number and decimal parts to multiply a decimal number by a whole number is commonplace in contexts where the calculation needs to be carried out mentally. This is an example of a vernacular strategy developed in the teeth of a practical problem, not learnt at school, and which occurs in many places throughout the world. In this context, banknotes and coins are auxiliary tools for the calculation.

THE MENTAL CALCULATION OF SQUARES

Given that $(n\pm 1)^2=n^2\pm 2n+1$, the square of a whole number can be mentally calculated using the square of the previous or following number:

$$31^2 = 30^2+2\cdot 30+1=900+60+1=961.$$
$$19^2 = 20^2-2\cdot 20+1=400-40+1=439.$$

Given that $n^2=a^2+n^2-a^2=a^2+(n+a)\cdot(n-a)$, it is also possible to calculate the square of a whole number based on its difference with another two, the product of which is more easily calculated:

$$19^2 = 1+(19^2-1^2)=1+(19+1)\cdot(19-1)=1+20\cdot 18=1+360=361.$$
$$37^2 = 9+(37^2-3^2)=9+(37+3)\cdot(37-3)=9+40\cdot 34=$$
$$=9+40\cdot(30+4)=9+40\cdot 30+40\cdot 4=9+1{,}200+160=1{,}369.$$

Bargaining: a commercial numeric strategy

Bargaining has always been a central activity in the field of commerce. In the West, it has all but disappeared, but in other parts of the world, the practice of haggling continues to live on in traditional and tourist markets.

The goal of bargaining is that the buyer and seller of an article reach an agreement that is satisfactory for both parties. It is common for the seller to initiate the process, beginning by stating the value the buyer must pay to acquire the article. It goes without saying that the first value will be exaggerated, sometimes highly exaggerated, meaning that the buyer must counter this first offer with a low price. However, not

excessively so, since in this case the seller might take offence and exclude the buyer, closing the bargaining process.

An unwritten rule of bargaining that occurs in traditional markets is that a good strategy is to find the way to bargain with the seller's first offer until reaching an agreement based on paying approximately half this price. However, this may change if the seller invites the buyer to make the first move.

It is most common to bargain for fixed quantities, although it may also be possible to haggle for discounts. If a discount of 5% is offered, it would be unreasonable to hope to be able to receive a discount of 50%, or rather half the price. Under such circumstances, bargaining until the discount is doubled and thus obtaining a discount of 10% may be regarded as a success. When we talk about discounts, the quantities to which they are applied are often large, and a small change in the percentage can represent a considerable sum of money. This is why it is often not so beneficial to bargain with percentages.

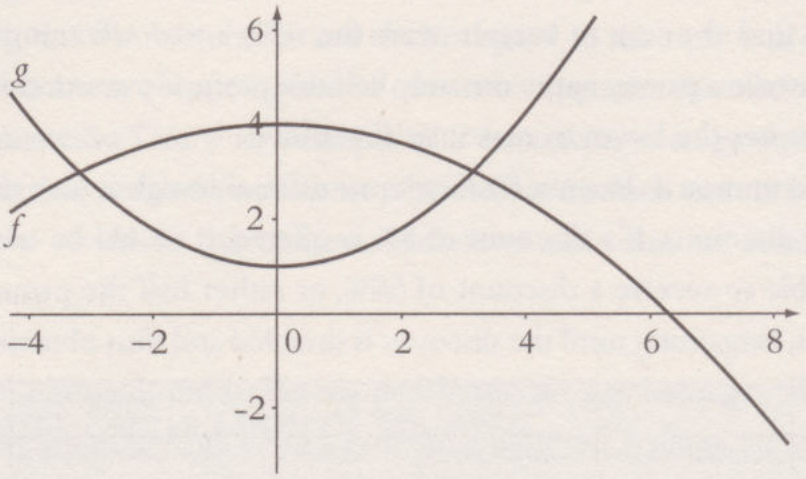
An initial mathematical model of bargaining may be linear, or rather, the variation in the prices offered by both parties follows a pattern of proportionality. This may be the simplest model, however we will soon realise it is inaccurate, since in the real world, the values offered do not increase or decrease at regular intervals, but instead the variations grow increasingly smaller as the two parties converge on an agreement.

A model based on a curved variation of offers would appear to be more suitable. The curve for the buyer, $C(x)$, will be increasing and concave. This means the prices offered by the buyer will increase, but the differences will grow smaller. For example, a series of values such as 20, 60, 100 and 140 conforms to the linear model, whereas the values 20, 50, 70 and 75 conform to the latter model. They are four increasing values, but the differences between them decrease. On the other hand, the curve for the seller, $V(x)$, will be decreasing, although the differences will grow smaller. Accepting a pattern in which the increase of $C(x)$ and the decrease of $V(x)$ are proportional to the last value that is offered we obtain parabolic curves, since the increase or decrease corresponds to the derivatives, $V'(x)$ and $C'(x)$, of each function. In the case of the buyer's curve, the derivative is positive ($C(x)$ is increasing), whereas for the seller it is negative ($V(x)$ is decreasing):

$$V'(x) = k \cdot x \ \ (\mathrm{k} < 0) \Rightarrow V(x) = k \cdot \frac{x^2}{2} + B.$$

$$C'(x) = m \cdot x \ \ (m > 0) \Rightarrow C(x) = m \cdot \frac{x^2}{2} + D.$$

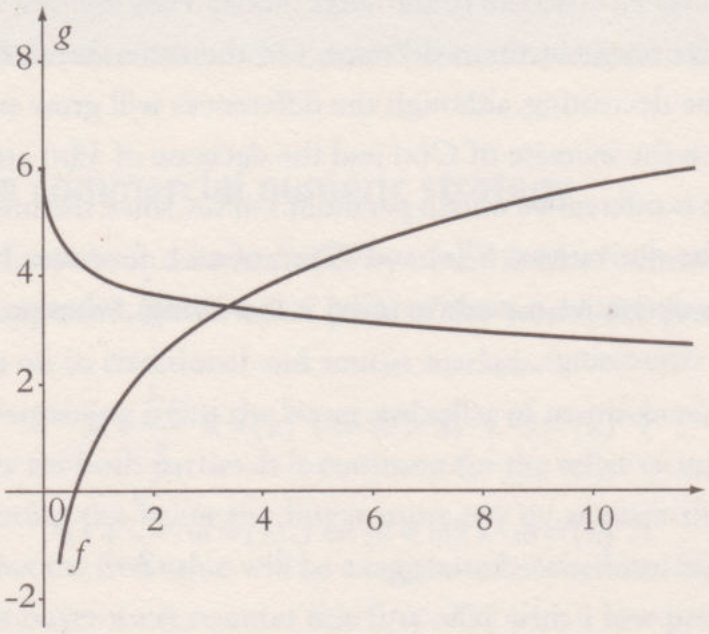
$V(0)=B$ is the starting price given by the seller. This gives two parabolas with different curves that meet at the point of agreement:



However, we do not know if both parties in the bargaining process think this way. In other words, if their offers are proportional to the last offer made. Is it perhaps the case that they might decide each offer should increase or decrease in a way that is inversely proportional to the difference separating it from the starting price? In this case, we have a new model, which is logarithmic, since this is the function that solves the differential equation that describes this thought process. Let V be the initial value of the seller:

$$C'(x)=\frac{k}{|x-V|} \Rightarrow C(x)=k\cdot\int\frac{1}{|x-V|}dx=k\cdot\ln|x-V|+K.$$

If the constant k is positive for the buyer, it is negative for the seller:



However, what happens in practice? Do people really make their offers based on similar, direct or inverse proportions? Obviously not. Let us consider the results of three real bargaining processes undertaken by me in various retail situations while on holiday. We are not talking about a small market or even a traditional market; the three cases are all from bargaining in shops. My offers were not even based on mathematical proportions or guides that had been previously devised; instead they were weighed up based on the considerations explained below.

Bargaining 1		Bargaining 2		Bargaining 3	
Seller	Buyer	Seller	Buyer	Seller	Buyer
45	20	80,000	40,000	350 (pm)	200
35	25	60,000	45,000	280	230
30	OK	50,000	OK	260	250
				OK	

Inside the shop, there were three articles with their prices marked, denoted by the abbreviation "pm" in the above table. This is often a sign of a fixed price. The price marked on the article I was interested in was 350. While I was considering the idea of asking if, as I had suspected, the prices were fixed, a saleswoman informed me she could offer me a discount.

I asked her how much. It's yours for 300, was the reply. It wasn't a large discount, leading me to deduce that, in spite of the fact that the prices were not fixed, perhaps they were extremely close to the real value. At any rate, the price I would pay would not be very low. I now had to think about how much to offer. I first thought about offering a price below 200, but this seemed too low. The largest number below 200 was 199, so I made an offer of 200, so that it would seem higher. The saleswoman replied with 280. I was a little disappointed with this offer, since it only differed from her previous one by 20. I guessed that we would finish close to 250, but I didn't want to approach this value too quickly. I offered 230, slightly over 225. She replied with 260. To conclude the process I told her that my final offer was 250. She insisted on 260, but I didn't back down. We agreed on 250.

When the bargaining process was finished, I asked the owner what margin she was prepared to accept, or rather, what was the minimum value she would accept based on the price marked on an article. She expressed her answer as a percentage: 25%. She explained that this was the policy in her shop, although, in other places, such as traditional markets,

the margin could be much higher. This being the case, buying an article with a marked price of 350 for 250 was relatively successful. The discount was over 28%.

Based on these practical results, it is possible to develop a new mathematical model of bargaining. There is something in the values in the table that is familiar to us, a sort of underlying equilibrium. Furthermore, they converge on the final agreed value. What is the pattern governing this equilibrium? We will formulate our hypothesis based on the fact that the equilibrium is governed by a situation whereby each offer is made by calculating the average of the last two values given. In other words, the sequence of numbers generated during the bargaining process, starting with the initial value x_0 offered by the seller and the initial value x_1 offered by the buyer conform to the general term:

$$x_{n+1} = \frac{x_n + x_{n-1}}{2},\ n \geq 1.$$

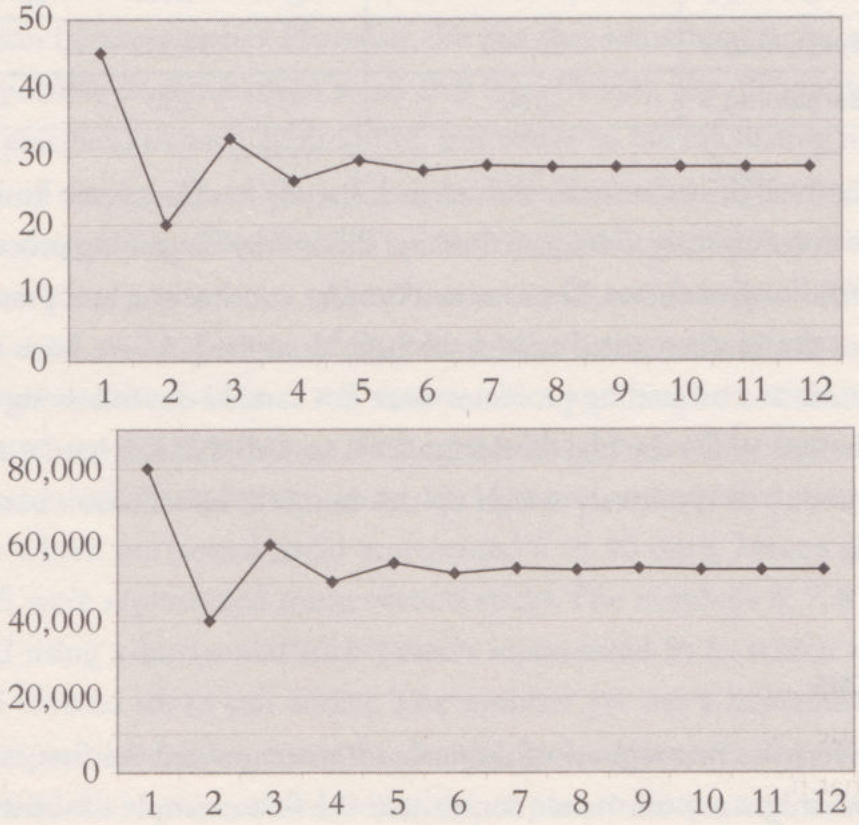
This is none other than the average, the arithmetic mean, of the last two values mentioned in the bargaining process. And this expression is extremely similar to the general term of the Fibonacci sequence. Does this model describe reality? Let's consider it by comparing the figures of the three previous bargaining processes with those of this model, which we shall call the average bargaining model:

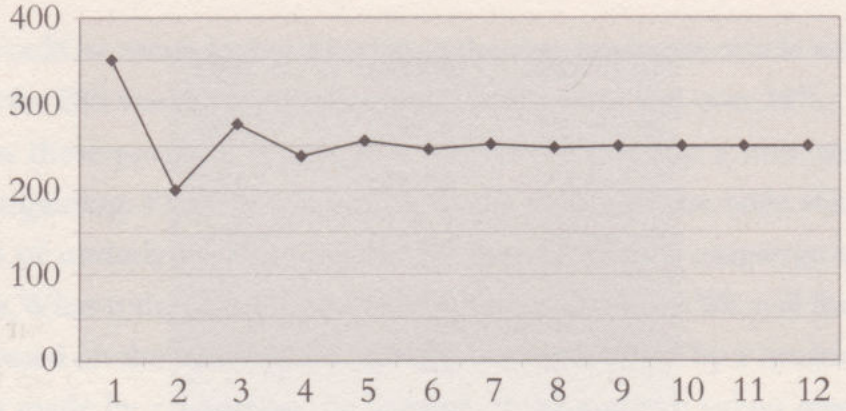
Bargaining 1		Bargaining 2		Bargaining 3	
Real life	Model average	Real life	Model average	Real life	Model average
45	45	80,000	80,000	350	350
20	20	40,000	40,000	300	300
35	32.5	60,000	60,000	280	275
25	26.2	45,000	50,000	230	237.5
30	29.3	50,000	52,500	260	256
				250	247

The similarity is striking. So, at least in the context of tourist gift shops, the average bargaining model is a good fit. It is now necessary to determine the value on which bargaining processes using this model converge. Put another way, what is the final value on which the bargaining processes of similar shops converge? Let's consider what happens if we take the initial values of the three previous bargaining processes:

45	80000	350
20	40000	200
32.5	60000	275
26.25	50000	237.5
29.375	55000	256.25
27.813	52500	246.88
28.594	53750	251.56
28.203	53125	249.22
28.398	53437.5	250.39
28.301	53281.3	249.8
28.35	53359.4	250.1
28.325	53320.3	249.95

How are these numbers related to the pairs of initial values for the bargaining processes (45,20), (80,000,40,000) and (350,200)? The corresponding graphs show their shapes are similar:





Studying the general term of this model clearly shows what happens. The limit X on which the series of values corresponding to the negotiation determined by the two initial values of the seller (x_0) and the buyer (x_1) converges is:

$$X = \frac{1 \cdot x_0 + 2 \cdot x_1}{3}.$$

Calculating X with the initial values for the three bargaining processes above shows the outcome:

	Agreed value	Limit X
Bargaining 1	30	28.333
Bargaining 2	50,000	53,333
Bargaining 3	250	250

Note that in all three cases, the fifth term is already so close to the limit that there is little sense in continuing to bargain. Perhaps this is why bargaining processes do not go beyond four or five values. Let us now consider another real and practical aspect that validates the mathematical model we have described. As we have mentioned, the aforementioned bargaining processes were not carried out following these rules. However, the model fits the real world situation so well that it is hard not to admire the human ability to intuitively weigh up this numeric information in search of an equilibrium.

The abacus

Our hands were the first register of numeric information and the first calculators in history. Some might say our hands constituted the first example of software. We can

use the fingers of one hand to count to 5, the fingers of two hands to count to 10, and we can use our toes to reach 20. However using our phalanges as units and our fingers as powers of 10, it is possible to count up to 10,000 million. It is possible that nobody has used this system on account of its somewhat unwieldy, impractical nature.

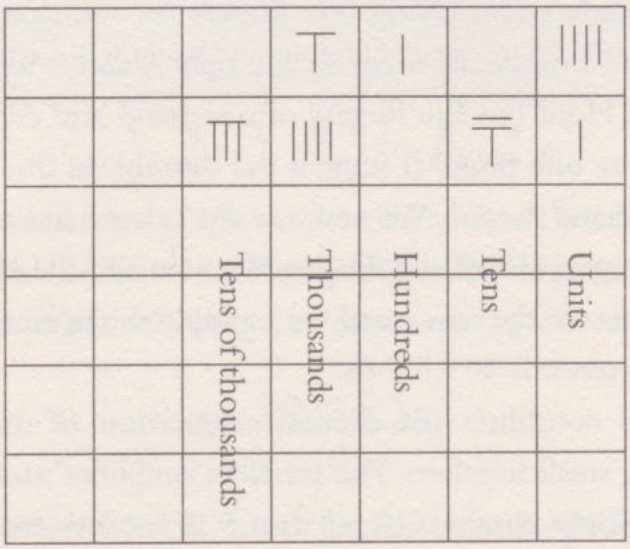
Beyond counting, our hands have been used by various cultures to calculate, especially for multiplication. Using our fingers for multiplication is common throughout Asia and Europe. In order to multiply 6 and 8 using our fingers, we proceed as follows. Hold out the fingers of one hand and count up to 6 – after number five we close one finger (I suggest the thumb). In this hand, we will now have 1 finger closed and 4 open. We now use the other hand to count to 8 in the same way, with 3 fingers closed and 2 open. We now add the closed fingers $(1+3)$ which gives 4 – these are the tens – and we multiply the open fingers $(4 \cdot 2)$, which gives the units. The result is $40+8=48$.

This procedure combines the mental calculation of simple addition and multiplication using small numbers. The problem simplifies matters since we never need to add or multiply numbers larger than 5. A modern mathematician would say that the multiplication of numbers less than or equal to 10 is reduced to a multiplication modulo 5. This system is used in everyday and even academic contexts in India, Indonesia, Iraq, Syria and north Africa. However, it is not very practical when applied to large numbers, although it is possible to expand the system in order to allow multiplication by any number. However, the fact that something is theoretically and practically possible does not mean to say that it is efficient, or sufficiently efficient to be used in practice. For such applications, it is better to use calculating instruments.

A verse of chapter 27 of *Tao te Ching* by Lao-Tsu states: "He who knows how to calculate does not use the chou". The chou was a calculating instrument formed of a wooden board and a series of bamboo sticks. Its use dates back to the 5th and 3rd centuries BC, making it one of the oldest calculating tools known.

The chou consisted of an 8×8 square tablet and bamboo sticks were placed in the cells to represent numbers. To begin with, the numbers were represented using the same number of sticks, up to 10. However, a simplified system was later adopted in which horizontal sticks represented 5 or 10 units. Hence, the numbers from 1 to 5 were represented using vertical sticks. The numbers 6, 7, 8 and 9 were represented using a horizontal stick (which represented 5) to which the required number of vertical sticks was added. The number 10 was a horizontal stick, and tens were represented using more horizontal sticks. However, for 60, 70, 80 and 90, the vertical sticks were added to the upper part to distinguish them from 6, 7,

8 and 9, hence the problem of positioning sticks to represent hundreds, thousands and larger powers of 10. The Chinese solved the problem using a board in which the lines of cells determined the positions of the digits. This gave rise to an empty cell to symbolise 0.



The numbers 6,104 and 84,071 represented on the chou.

Multiplication was carried out by combining the mental calculation of small products and sums represented on the board. Just like in the African markets we mentioned earlier on, its essence was decimal decomposition and the implicit use of a property that would later receive the name distributive. To multiply 285 by 43 for example, an empty line would be left between the lines of both numbers for the intermediate calculations. The mental process was as follows:

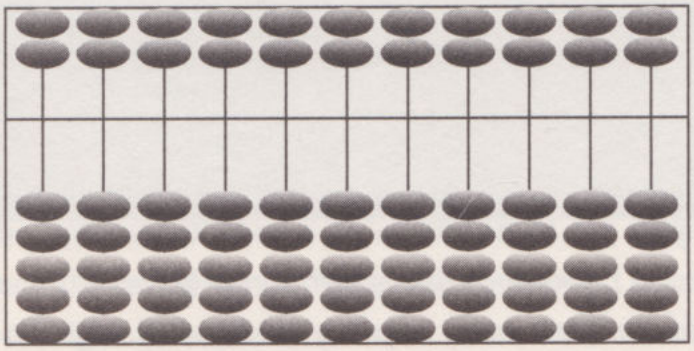
285 × 43			
Stages	Mental calculation	Result	Positional value
1	4×2	8	8,000
2	4×8	32	3,200
3	4×5	20	200
4	3×2	6	600
5	3×8	24	240
6	3×5	15	15

Hence, it was based on the decomposition of 285 and 43 into hundreds, tens and units:

$$\begin{aligned} 285 \cdot 43 &= (200+80+5)\cdot(40+3) = \\ &= 40\cdot 200+40\cdot 80+40\cdot 5+3\cdot 200+3\cdot 80+4\cdot 5 = \\ &= 12{,}255. \end{aligned}$$

The board was also used to solve equations and systems. Furthermore, the written numeric notation using bars has also been attributed to the chou. There are also those who believe it to be a predecessor of the abacus, which was invented many years later, towards the 14th century.

In spite of being a very old instrument, the day-to-day use of the abacus throughout the world is surprising, especially in countries in Southeast Asia (such as Singapore and Thailand) and East Asia (such as China, Korea and Japan). In Japan it is called a *soroban*. The abacus is an elongated rectangle that is generally made from wood, with a horizontal bar and a number of vertical bars that hold seven small wooden balls. Two of these are above the horizontal bar in the middle; the other five are below it. The number of vertical bars can vary between eight and twenty, and may even be higher.



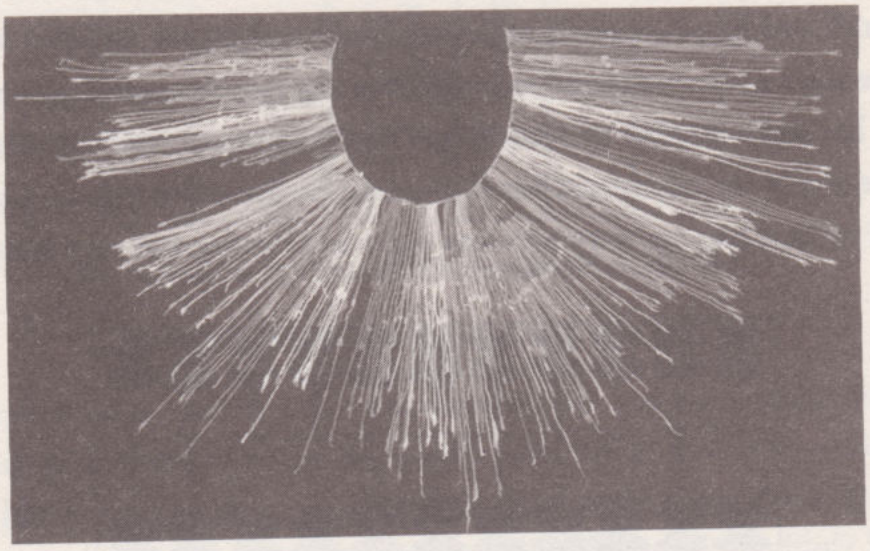
Given that each bar represents the units from 0 to 10, and that each of these corresponds to a power of 10, on an abacus with 20 it is possible to represent numbers up to $10^{20}-1$.

Not everywhere has bamboo or wood to build calculating tools such as the chou or the abacus. We have already seen how pebbles were used to represent digits and numbers thousands of years ago in Sumeria. In America, the Mayan numbering system is similar to the Chinese system and used stone as a basis for measurements. Moreover, the Mayans had a symbol for zero. Much further south on the American continent, we can find another unusual calculating instrument that is quite different from the

chou and the abacus. Just like these, it was easily transportable, not only because of its size, but also due to its flexibility. Aside from the human body, the Inca quipus were the first soft tools for recording numeric information.

The quipu

Quipus are bundles of strings used by the Incas to record their accounts. Analysing the specimens that have been preserved reveals how a record of numeric information was kept. An abacus consists of pieces of wood threaded on a bar or wire; the quipu did not use threaded counters, but knots instead. The digit or number represented by a knot depended on its position on the string and its colour.



Inca quipu (photograph: Claus Ableiter).

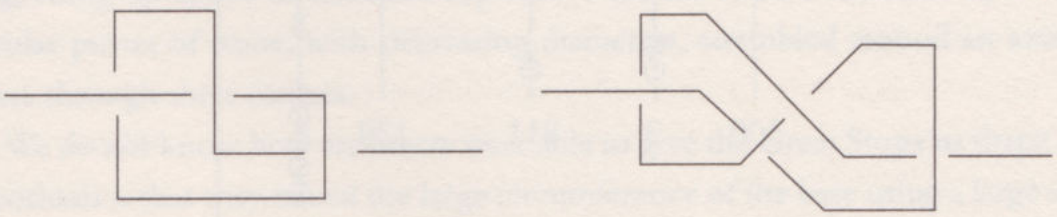
In general, quipus were made from wool or cotton. Strings of all kinds were extremely important to the Incas, because they could be used for all matters relevant to their life, such as building bridges and paying taxes. It is thought that quipus were used for recording accounting information related to censuses and harvests, although they have still not been fully deciphered and as such it has not been possible to explain all the uncertainties surrounding their meaning.

On a quipu, the way in which the knots are tied, as well as their colour and position relative to other knots and strings, is important. When extended, a quipu forms a line from which a series of strands hang. Brushing it separates each strand so they can be seen easily and shows the configurations of the whole piece so we can attempt to guess its meaning.

The Incas did not have a system for writing, and the quipus are the closest we can find among the relics of their culture. It is possible that some quipus referred to historical and social events from life and that their meaning is not purely mathematical.

A quipu is composed of a main string, which is thicker than the others, and from which a series of thinner strings hang, and on which the knots are tied. These secondary strings can branch out into third, fourth and higher orders, thus showing the tree-based structure of the piece. Their number can vary from just a few all the way to hundreds or thousands. The secondary strings may branch from the main string in different directions. When the quipu is stretched out on a surface so that its main string forms a horizontal axis, some of the secondary strings point upward and some point downward. The distance separating the points of connection of the strings from the main string makes it possible to distinguish them. The same holds for secondary strings and those of other orders. They are also distinguished by their colour, since a quipu could be monochrome or multicoloured. Furthermore, just as the knots often represented numbers, the colours could refer to the contexts to which they belonged, such as distinguishing between the products or groups of people to which they made reference.

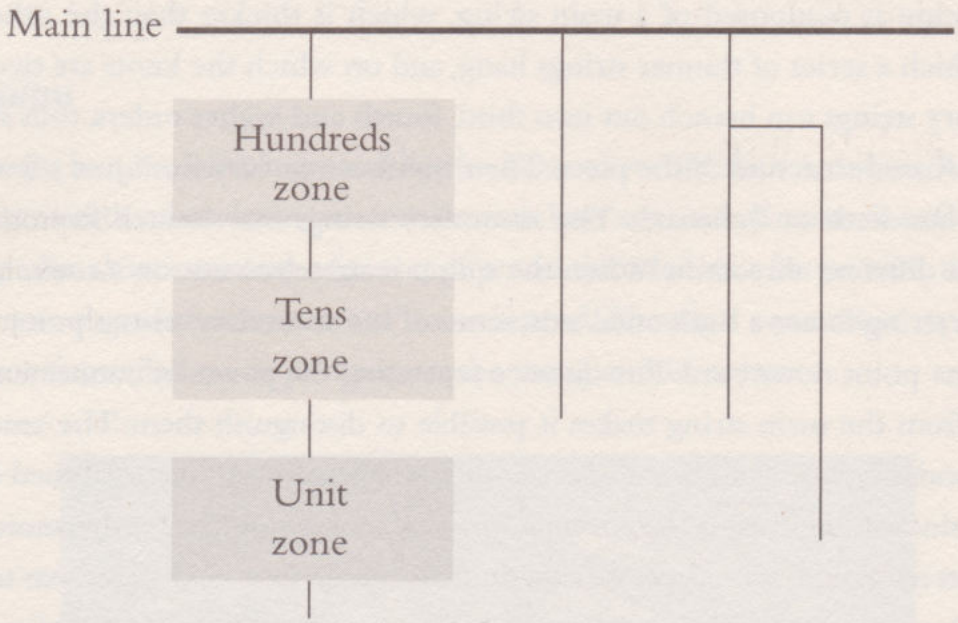
The quipu numbering system is decimal and positional, like ours, and is expressed using knots. Marcia Ascher, a mathematician from the United States and an ethnomathematical researcher has studied many quipus and classified their knots into three types: simple, compound and figure-of-eight (sometimes known as a reef knot). The simple knot (or hitch) is the most straightforward and is known by everyone. The compound knot is an extension of the simple knot. If, in the case of the former, the string has one turn, the compound knot has two or more. For the figure-of-eight knot, there are two turns, each in different directions. Here another type of special knot should be included – the absent knot which is interpreted as a zero.



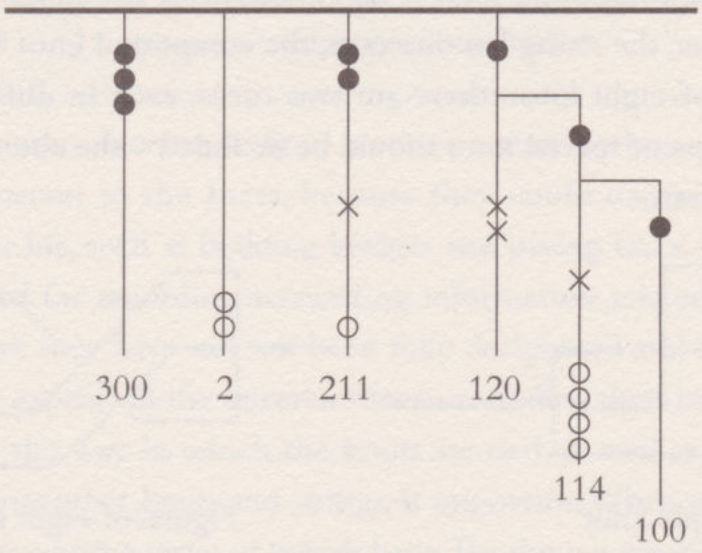
Simple knot Figure-of-eight knot

Let's adapt the symbols used by Ascher to represent these knots: a thick point for the simple knot and a small cross for the compound knot. However, instead of

using the letter E (for *eight*) for the figure-of-eight knot, we shall use the letter O. In each string, the knots are grouped into intervals. Each interval is related to the corresponding whole number power of 10, counted from the end.



Simple knots are not used at the ends. To begin with, it is possible to use compound and figure-of-eight knots, but it would be somewhat meaningless to represent a unit using a compound knot. This is why a figure-of-eight knot is used. Hence the units, located at the ends of the strings, are expressed using a figure-of-eight knot. Using the conventions we have described, it is possible to represent various knots on an imaginary quipu:



The quipu is an encoding and positional decimal numbering instrument, although it is not known if it was used as a tool for calculation.

Chapter 3

Mathematics for the Gods

Asian architecture

During the thousand years of the Middle Ages, hardly any scientific progress was made in Europe. Only the Italian Renaissance and the great ocean explorations succeeded in rousing the continent from its slumbers. Thanks to these journeys, Europeans became aware that there were many things beyond their continent. There were goods and riches, but there was also something else. There were other peoples and cultures, other beliefs and ways of life. There were unknown vegetables and plants that would enrich the diets of Europeans. There were textiles, designs and architecture. And hence, there was mathematical thought.

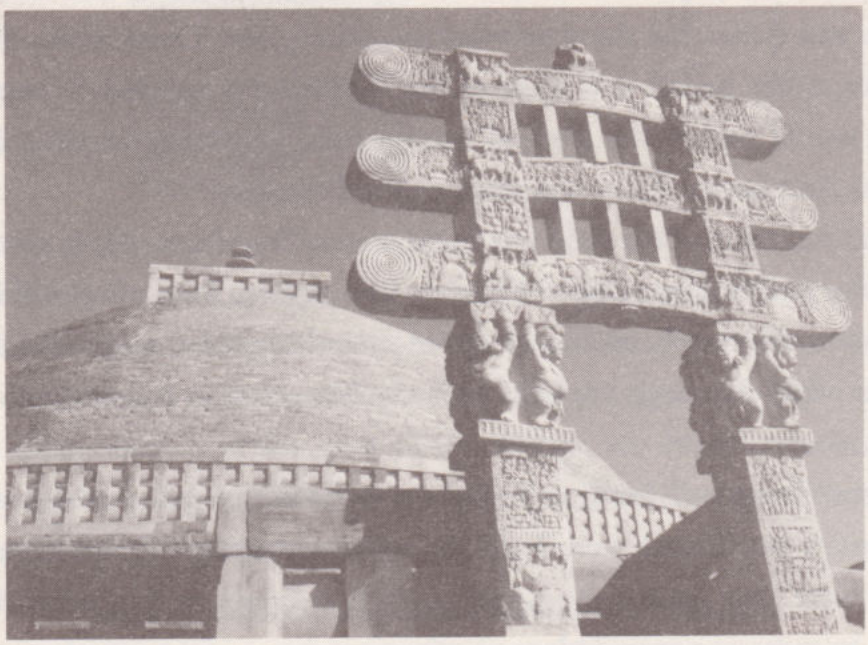
Asiatic architecture of the time was based around Buddhism. More than a religion, Buddhism is a philosophy of life centred on four aspects. First, existence is suffering; second, if you suffer it is because you desire something; third, to avoid suffering it is necessary to avoid desire; and fourth, desire is eliminated by following the eightfold path of Buddha.

The Great Stupa of Sanchi in India is a Buddhist religious construction from the 1st century BC. Stupas were built for various purposes. They were originally tombs, but were later used to house relics, such as bones or fragments of the body of Buddha, to indicate a sacred place or commemorate an important event. Pilgrims had to walk around the perimeter of the stupa in a clockwise direction.

The structure of the Great Stupa is hemispherical and has a diameter of 40 m. Like all stupas, it is crowned by a cube or die, the sides of which are almost 6 m long, rising up above its flattened top. Above the die is a dome, made up of three circular pieces of stone, with decreasing diameters, assembled around an axis that passes through their centres.

We do not know how architects were able to give the Great Stupa its shape. One hypothesis is that they traced the large circumference of the base using a large string for the radius. However, how could they have obtained the hemispherical curvature of the dome? We say hemisphere, but is it really that shape? The walls of a hemisphere touch the ground at a right angle. This is not the case with the stupa. How did they give the die on top its square shape? This assumes having the knowledge required to

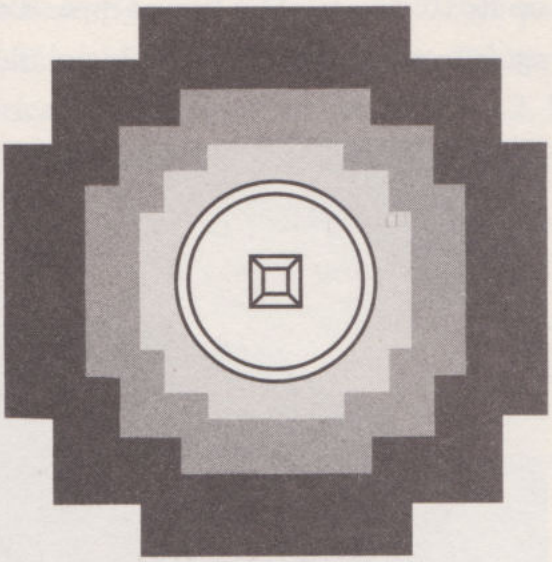
create right angles. Did they use the same method as the ancient Egyptians? At that time, it was already known that a triangle whose sides were proportional to 3, 4 and 5 had a right angle. The use of this property for marking right angles on the ground is commonplace in modern-day construction, and it has been documented in places as distant as Argentina and Sweden. Another practical way to construct a right angle is to draw a triangle with two equal sides and connect the vertex that joins them to the midpoint of the base. This line segment determines the height of the triangle. The procedure is similar to the one explained in Chapter 1 for constructing the rectangle for the bases of the Egyptian pyramids.



The Great Stupa of Sanchi, in Madhya Pradesh, India (photograph: Tom Maloney).

At any rate, the structure of the stupa evolved into more sophisticated versions. The stupa of Bodhnath in Nepal is also hemispherical, but is seated upon a base that represents a mandala. Mandalas are geometric and astrological representations based on the idea of concentricity. Their structure is often circular or square and is made up of irregular concentric polygons derived from the square. This forms the base of this stupa, which is also crowned by a die.

In contrast to the Great Stupa of Sanchi, the Bodhnath stupa culminates in a pyramidal stepped dome, made up of a series of 13 overlapping squares the sides of which decrease in length. Each of these 13 levels represents a stage on the journey to Nirvana.



Left, view of the stupa of Bodhnath in Nepal (photograph: MAP). Above, diagram of the floor plan of the building.

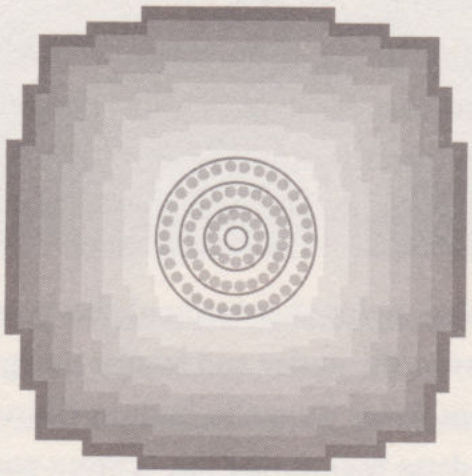
The lantern tower on the die characterises stupas and dagabas, which are similar monuments. Regardless of whether they are circular, such as the stupa of Sanchi, pyramidal, such as the stupa of Bodhnath, or conical, such as the dagaba of Anuradhapura, in Sri Lanka, they are all geometric and their diameter always increases towards the sky.

Perhaps it was this final point that inspired the construction of pagodas, in which the circle gives way to the square and regular polygons. Of Nepalese origins, these temples with various floors were also built in China and Japan. The Dayanta pagoda in Xian (China) dates back to the 7th century and has seven square floors. The Fogongai pagoda in Yingxian, dates back to the 11th century and also has seven floors, although this time they are octagonal.

The architecture of the temple of Borobudur, on the island of Java (Indonesia) fulfils three purposes. It is at once a stupa, a mandala and a replica of Mount Meru – the place that was believed to be the home of the gods. This makes it a Hindu and Buddhist temple at the same time. Completed in the 9th century, its shape certainly recalls that of the stupa, since the temple is built in a series of levels to produce a hemisphere. However, in contrast to the stupas, this shape is not the fruit of a single

hemispherical structure, but the product of the accumulation of smaller structures distributed over terraces. It is also a mandala on account of the design that makes up its 10 terraces. The first of these is still buried. The 10 levels are crowned with a stupa in the shape of a bell which holds a statue of Buddha.

The stupa of stupas in Borobudur, Java, Indonesia (photograph: Gunawan Kartapranata). Below: Floor plan of the temple.



By ascending these 10 levels, the pilgrim is following the path to Nirvana by means of a dialogue of circles and squares that takes place on the outside, since this sculptural mountain does not have an inside. The only ritual that can be carried out is to walk around the monument, which has a square base with sides of 100 m.

The numbers present at Borobudur are surprising on account of their multiplicative structure. To begin with, instead of building a large stupa, small stupas are multiplied throughout the three upper levels. This is done in circles that ascend towards the top

stupa in series of 32, 24 and 16. Each of these contains a statue of Buddha, and in addition to these, there are another 504 images of the founder of Buddhism.

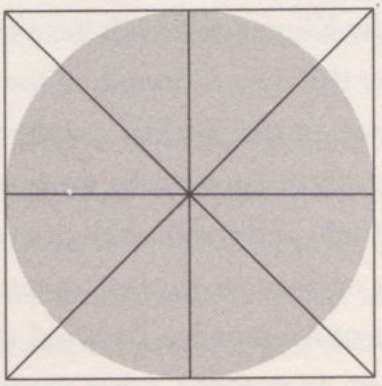
The decorations carved into the walls of the levels occur in series of 120, 128 and 72. There are 2,700 sculpted panels in total. If geometrically speaking, the dialogue focused on the circle and the square, numerically it is focused on the numbers 2, 3, 5 and 7, using them as bases and exponents of a power:

$$120 = 2^3 \cdot 3 \cdot 5$$
$$128 = 2^7$$
$$72 = 2^3 \cdot 3^2$$
$$504 = 2^3 \cdot 3^2 \cdot 7.$$

Some of these numbers can also be broken down into a product of consecutive natural numbers:

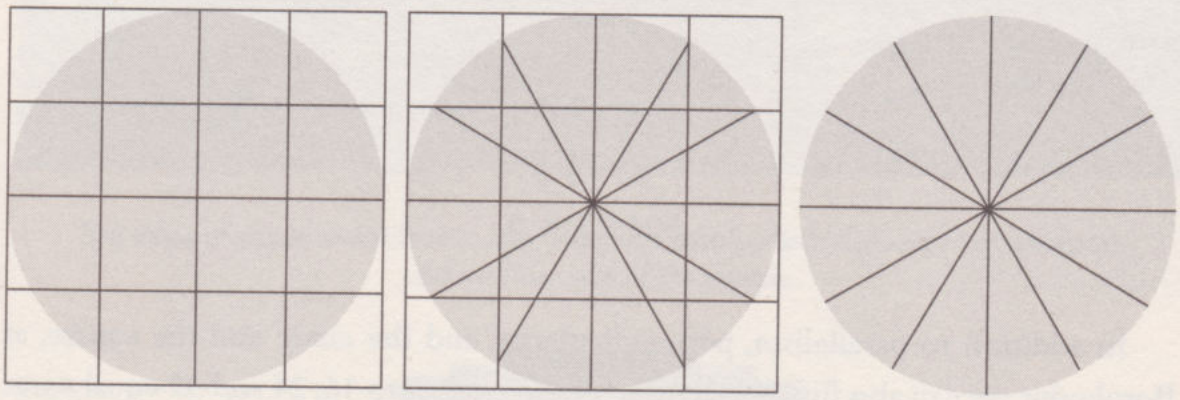
$$120 = 4 \cdot 5 \cdot 6$$
$$72 = 8 \cdot 9$$
$$504 = 7 \cdot 8 \cdot 9.$$

In addition to parallelism, perpendicularity, and the circle and the square, at Borobudur we can also find the division of the circle into 16, 24 and 32 equal parts. Inscribing a circle in a square and drawing either the diagonals or the lines to connect the bisector of the sides, the circle is divided into 4 equal parts. The bisectors and the diagonals divide the circle into 8 equal parts. Were these the geometric points of reference used to arrange the architectural elements? In this case, it would be enough to add more bisectors, albeit approximately, to divide it into 16 parts, and repeat the same process to give 32.



Division of the square and the circle into 2, 4 and 8 equal parts.

Dividing the circle into 24 equal parts means being able to divide it into 3 or a multiple thereof, such as 6 or 12. There is a simple technique to divide it into 12 parts without knowledge of trigonometry. Did the architects of the 9th century who built Borobudur use this technique? It consists of inscribing a circle in a square, which can be done by drawing the square that circumscribes the circle by means of four lines tangential to the circle and perpendicular to each other. The sides of the square are then divided into four equal parts and the corresponding grid is drawn. Finally, each point of intersection of the grid and the circumference is joined to the point diametrically opposite to give a circle that is divided into 12 identical sectors. From here it is enough to draw the bisectors of each sector to divide the circle into 24 equal parts:



A division of the circle into 12 equal parts.

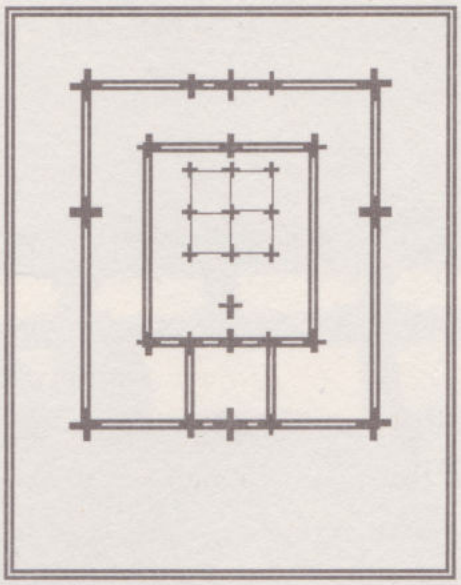
However, this model is appropriate if we are using a pen and paper. It is probable that the 24 stupas of the level of the temple of Borobudur are spaced equidistantly by measuring the length of the circumference and then dividing it into 24 parts, thinking of the line and not the circle.

The temple of Angkor Wat, in Cambodia, which dates back to the 12th century, represents the high point of the Khmer culture. It lies a few kilometres to the north of the city of Siem Reap. *Angkor Wat* means 'temple of the capital'. Its size makes it one of the largest temple complexes in the world. Its architecture is based on the square and the rectangle. It was originally designed to be the tomb of King Suryavarnam II, as well as for worshipping the Hindu god Vishnu, although it is believed to have a cosmic and iconographic symbolism on account its size, orientation, shape and sculptures.

The fact that the temple of Angkor has survived the passing of time is due to the fact it was built from stone. In contrast, other temples, such as the first pagodas, were built from wood and disappeared in the jungle. The rectangular area enclosed

by the main construction of Angkor Wat is a rectangle that measures 341 m long by 270 m wide.

The temple of Angkor Wat, in Cambodia (photograph: Bjørn Christian Tørrisen).



Floor plan of Angkor Wat (Cambodia).

Given that its primary function was to be as a tomb, like the great majority of such constructions around the world Angkor Wat was built looking towards the West. The temple illustrates the Hindu system of cosmology, with Mount Meru in the centre of a series of concentric continents surrounded by the sea. Entering the temple on 21 June, the central tower indicates the path the Sun will follow in the sky. In Indian astronomy, this is defined as new year's day. The distance separating the entrance of the central altar of the temple is 1,728 hat, the Khmer unit of measurement, which

corresponds to the 1,728 years of the first golden age of the universe, also according to the Hindu system. Hence, the temple of Angkor Wat represents evidence of Khmer knowledge of the time. Beyond the art of its sculptural decorations, in mathematical terms it represents a majestic example that includes design, symmetry, parallelism, perpendicularity, the rectangle and the circle, measurement and number.

Buddhism extended throughout Asia from India and through China, to reach Japan in the 6th century. The country already had an indigenous religion based on nature, which was given the name Shintoism, to distinguish it from the new religion. A Japanese person was not required to choose between both religions, and the majority of Japanese people are Shintoist and Buddhist at the same time. While the former is related to practical and real-life matters (harvests, the economy, success and work), the latter is related to loftier matters, such as ethics and the fate of the soul.

The majority of districts in Japan have Shintoist shrines and Buddhist temples. They can be easily told apart by their entrances. The entrance to a Shintoist shrine is a *torii*, a construction consisting of two vertical columns and two crosspieces at the top. Traditionally made from wood, it is the door to enter the shrine and is painted bright red.

Entrance to the grounds of the Shintoist shrine of Fushimi-Inari Taisha in Kyoto (photograph: MAP).

The composition of the torii may be more complex depending on its size. The imposing structure of the large torii that marks the entrance to the shrine of Itsukushima-Jinja, on the small island of Itsukushima, is made up of three vertical

planes, one of which defines the crossbeam for the other two parallel planes. Adding to the set of the horizontal plane π of the sea from which it emerges, the structure comprises four planes π_1, π_2, π_3 and π, whose ratios of parallelism and perpendicularity are:

$$\pi_1, \pi_2, \pi_3 \perp \pi$$
$$\pi_1 \| \pi_3 \perp \pi_2$$

All this is defined by 12 enormous trunks or segments of wood.

The large torii of the shrine of Itsukushima-Jinja (photograph: MAP).

To access the Fushimi-Inari Taisha complex of Shintoist shrines in the outskirts of Kyoto, it is necessary to pass through more than one thousand toriis that form part of the 4 km long path around the hill. In some sections, they are separated by scarcely a few millimetres. This series of doors, or rather planes, creates a three-dimensional space, a prism of curved parallel walls that ascend with the formation of the hill. The series ends when it reaches the main shrine.

Vernacular architecture on the new continent

The Aztec culture inhabited Central America between the 1st and 6th centuries. Teotihuacan, its ceremonial centre, was a gridded city the layout of which was determined astronomically, so that the city was a model of the sky and the movement of the heavenly bodies. The central avenue connected large stepped pyramids at

the top of which, accessed by long stairways, stood temples where bloody human sacrifices were made.

The Aztec pyramids were constructions with a square base and four levels. The largest and oldest had sides of 213 m and was more than 60 m high, being positioned so that it marked the axis of the Sun on the solstice. With the exception of this example, which has sloping sides, the faces of the four levels of the pyramids of Teotihuacan are vertical.

The Mayan culture was contemporaneous with the Aztec culture, but lasted longer. Like the Aztecs, the positioning of Mayan buildings conforms to astronomical observations. It was the first American culture to discover the vaulted technique. Its pyramids were also stepped and reached heights of 70 m. However, the bases were not perfect squares. The pyramid known as the 'Castle', in Chichen Itza, has a square base and nine stepped levels that end in a die-shaped temple. Four sets of stairways with 91 steps, one set per face, lead to the temple. Curiously, $4 \cdot 91 = 364$ is almost the number of days in a year. Some have interpreted these steps as a model of a calendar.

Upon reaching Peru, sometime around the year 1500, the conquistadors came into contact with the Inca Empire, which at the time extended along almost the entire length of the Andes Mountains. Inca technology, perhaps on account of the fact that it was a culture with more practical interests than others, was the most advanced in the Americas. They produced textiles, made alloys of gold and copper prior to the 10th century, and had irrigation and terrace-based agriculture systems.

In the 13th century, the capital of the Inca Empire was Cuzco, a city located on the 'royal road of the mountains', surrounded by walls some 6,000 km long that connected the points of the empire and were a source of great amazement to the Spanish. Their polygonal masonry of large blocks aligned with millimetre precision, and the rectangular and circular shapes of its buildings with trapezoid windows and doors constitute a hallmark of this culture.

The Inca architecture does not exhibit a special predilection for right angles. The frames of the windows and doors are trapezoidal. Nor does the masonry of its walls conform to a grid-based structure with right angles; on the contrary, it is characterised by all sorts of angles and pieces that fit so precisely together they have become a symbol of cultural identity. However, this structure represents an extraordinary exhibition of parallelism.

Every face of every stone is perfectly moulded to that of its neighbour, as if the two regular faces they possessed had been filed down by rubbing against each other until becoming flat and smooth, whereby only the segment or line of contact was

visible. Although we don't see them, the faces and edges of these large blocks are parallel, and this parallelism would appear to be the fruit of a practical action.

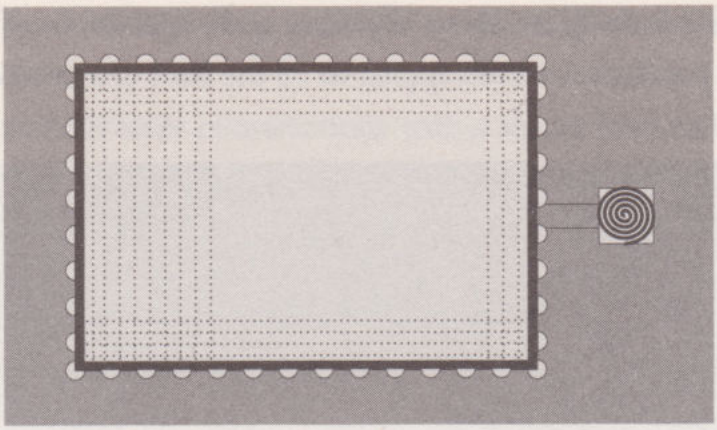
Inca wall in Cuzco, Peru (photograph: Martin St-Amant).

Islamic architecture

The Great Mosque of Samarra (Iraq) was built in the 9th century, and for many centuries was the tallest in the world. Today, all that remains is the rectangular perimeter that enclosed it, the walls of which it was made up and the spectacular 50-m high spiral minaret. The enclosure of the mosque, like others of its time, was designed so that its dimensions were in a proportion of 3:2. Along the length of its walls, and on the outside, 44 cylindrical columns were arranged as buttresses. The side ones had a semicircular base, whereas the bases of those used for the four corners were three quarters of a circle.

In this mosque, we can find geometric elements that were common to other temples that have been mentioned, such as the rectangle, the square base of the minaret, and the parallelism and perpendicularity of the columns of the courtyard. However, in this case the circle and the square are combined to create a spiral composition.

The spiral was already present as a symbolic element in the stupas (see the image at the start of this chapter), but the minaret of the mosque of Samarra is based on a spiral that ascends towards heaven. Its spiral staircase reaches the top in seven turns around the axis. The slope of the ascent is not constant, with the second last turn being the steepest. Nor is the structure cylindrical; instead it is conical, closing towards the vertex.



Floor plan of the mosque of Samarra (Iraq).

Climbing the spiral staircase of the minaret of Samarra presents a problem. Since there are no handrails, climbing the outer part of the steps is risky and may cause some people to suffer from vertigo. Staying on the inside is safest. However, here we come up against a curious aspect of spiral staircases: in contrast to normal stairs, which have a constant slope, regardless of which side we climb up, the gradient of spiral staircases varies, even though all the steps are identical. This is caused by the fact that the steps are wider towards the outer edge. Each step ascends by the same height (vertical distance), but the horizontal distance travelled on the outer part is greater, making the quotient of the vertical ascent and the horizontal distance smaller and the gradient smaller. In contrast, the length of the distance travelled is greater. Climbing the inner part of the spiral staircase means the distance is shorter, although the climb is harder; climbing the outer part means a longer distance, but requires less effort.

Offerings worthy of the divinity

Thus far, we have discussed mathematical ideas related to religious architecture. Let us now turn our attention to an issue that is important in old religions and that directly involves their followers. Believers throughout the world address their god or gods by means of prayers, and in the majority of cases, objects, food and gifts are offered to placate their wrath, make peace with demons or evil spirits, or request good fortune in life.

If there is one place in the world where practically all aspects of life are governed by religion, it is the island of Bali in Indonesia. In contrast to the Islamic religion that predominates in the rest of the country, Bali inherited the Hindu religion from

India. Thousands of temples and altars, large and small, can be found everywhere. No house is complete without one and they may be found in holy sites and sometimes also at dangerous places such as crossroads or motorways.

The day begins with the *sesajen*. This is a brief ritual that is generally undertaken by women three times a day before meals. In addition to prayers, a series of offerings in the form of food from the previous day are placed on the ground, beside household temples, at the entrances to houses and crossroads. These are referred to as *cana* (pronounced 'chana') and consist of small portions of cooked rice, scraps of meat, biscuits, flowers, incense and holy water. It is hoped that the gods who inhabit all things decide to try a mouthful. Birds make light work of whatever the gods leave behind.

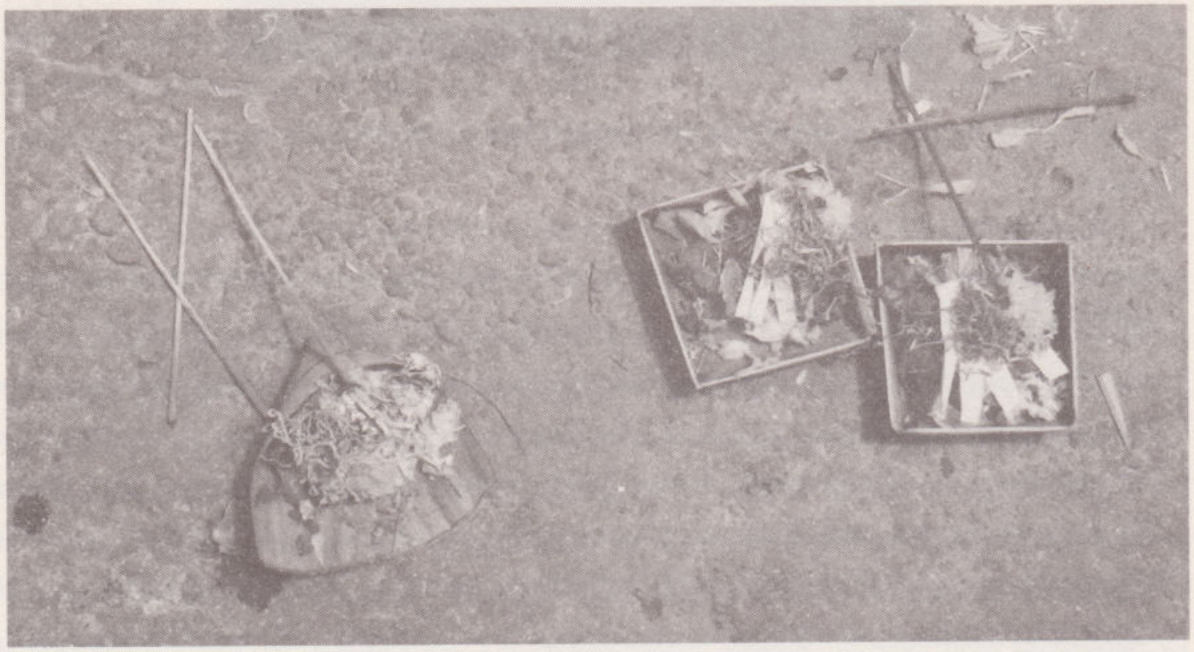
MONEY AND MATHEMATICS

There is one thing that is essential for all banknotes throughout the world – they must be extremely hard to forge. For this reason, metal materials have been incorporated into the paper to form hidden identification marks. In the banknote for one Qatari dinar, we can see chains of polygonal knots interlocking in knots with a symmetry of order eight, a sailing ship that is reproduced proportionally at different sizes. There is symmetry in the arches and the pillars and an irregular white hexagon obtained by cutting the corners of a square.

Similarly, coins from Brunei reproduce spiral designs of the peoples that inhabit the jungles of Borneo. The first thing a tourist will come across when they arrive is the mathematics documented by the currency.

The back of a Qatari 1 dinar note and a 10 sen coin from Brunei.

An offering for a god cannot be made in just any old way. The gods deserve respect, and this must be made explicit not just when communicating with them, but also in the way in which offerings are presented and the recipients that contain them. Hence, great care is taken in the preparation of the containers for these offerings. The tender leaves of palm and banana trees are used, having previously been cut to a specific geometric shape and size.



Cana *for a* sesajen *in Bali, Indonesia (photograph: MAP).*

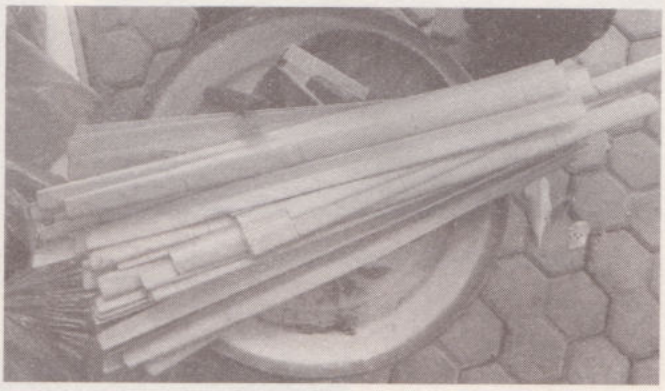
The shapes of the receptacles may also vary considerably, although the most common shapes are shown in the photograph. They are not the product of chance. On a daily basis, women of all ages can be seen folding and weaving leaves to make the boxes and envelopes in which the offerings are placed. Upon seeing them, the question arises of how the creator of the small boxes can be sure they are square and how they obtain the right angles for the envelope.

It is not necessary to make speculative hypotheses about how these geometric shapes are made because we have that information direct from the creators. The procedure followed by a woman from Bali to create these square receptacles is described below, accompanied by illustrations.

The process begins by cutting the tender leaves of a palm tree into strips of a similar width, determined by the veins. Taking the distance between the end of the index finger and the thumb as a unit of measurement, four consecutive marks are made on the leaf. The palm leaf is then folded so the last mark coincides with the first. Various

pieces of banana leaves that have been previously cut to the same measurement are assembled to create a larger piece that is inserted into the quadrilateral formed by the palm leaves. The square is now ready to be filled with offerings.

1. The unit of measurement.



2. Palm leaf rods marked with four units.

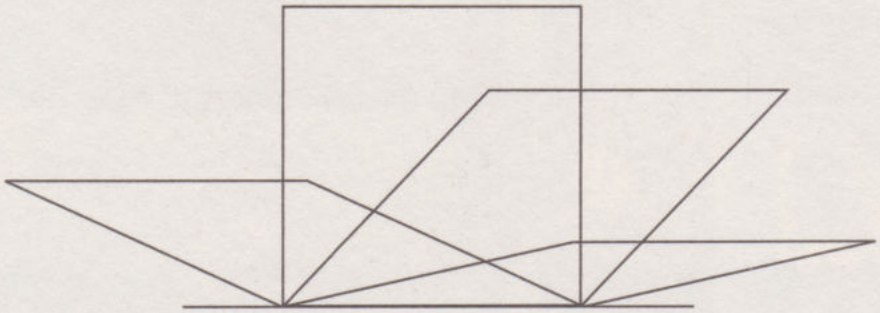
3. Folding the palm rod.

4. The length of the piece of banana leaf is one unit.

5. A number of pieces of banana leaf are assembled to form the base of the receptacle.

6. Once the bottom has been assembled, the object is complete (photographs: MAP).

The maker knows she has made a square because she can see it with her own eyes, and also because she has made it by folding a segment with four equal parts. This guarantees the four sides are equal, but does not guarantee the angles. In fact, the folded quadrilateral is a rhombus. The square is made by inserting the piece of banana leaf. Since this is of the same length as the side of the square, the height of the rhombus will be identical to the side. Hence, the box is square. Out of the infinite number of possible rhombuses (quadrilaterals with equal sides), just one is a square, and furthermore this has the largest area:



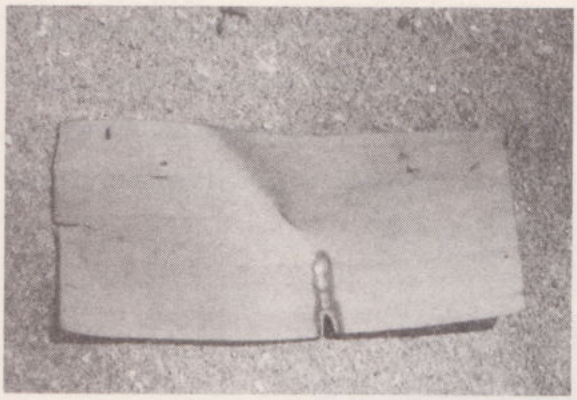
Just some examples of the infinite number of rhombuses whose sides are equal.

A Euclidean mathematical analysis might lead us to believe that the practice is not sufficient to guarantee the conclusion. From this perspective, the woman puts the application of a theory into practice: *a rhombus with a height that is equal to its sides is a square.*

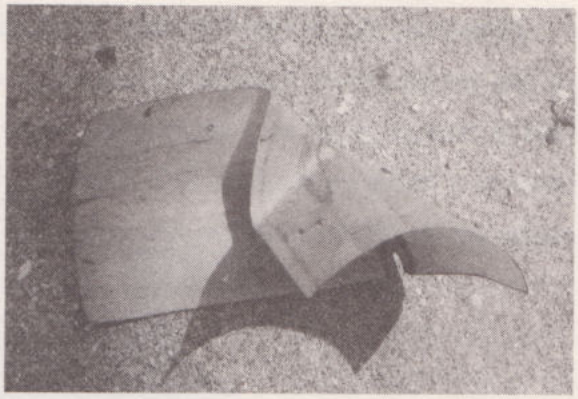
The proof is simple. Two adjacent sides of the rhombus can form two sides of a right-angle triangle. However, it will only be a square if those sides form the catheti, and are thus perpendicular to each other.

Correctly prepared smaller versions of the same rods are used to create a range of designs to be included in the offerings. They are also geometric and some reflect the shapes of flowers. They are created by means of folds and turns of the same extent.

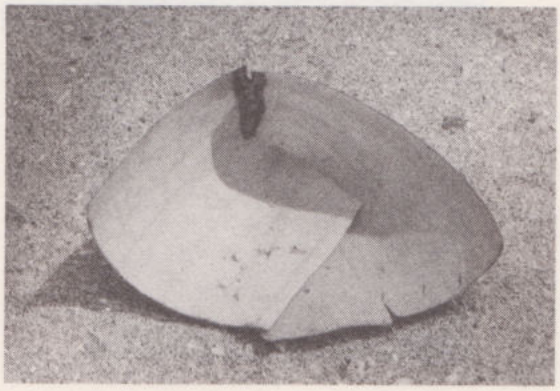
The creation of an offering vessel is shown in the following images. It was created from a rectangular piece of banana leaf. Its length must be approximately double its width. Its centre and bisectors are marked and it is folded so the bottom corners overlap. This means the upper profile is bent into a curve and creates the hollow into which the offerings are placed.



Mediatrix marked on a rectangle of banana leaf with an aspect of 2:1.

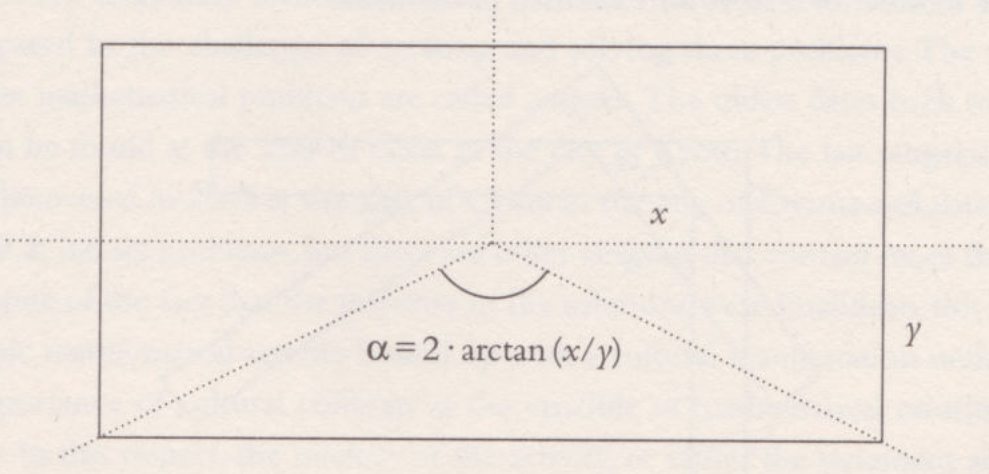


First diagonal fold of the rectangle.



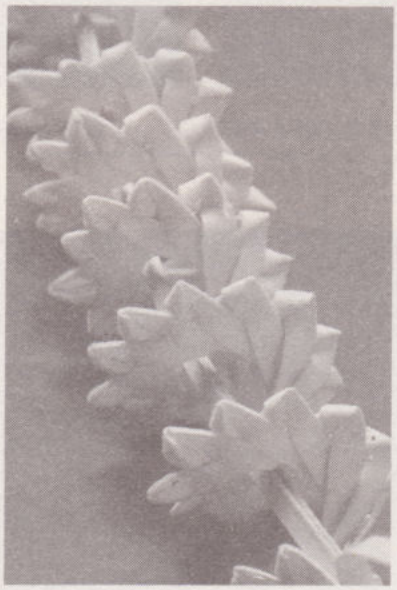
The second diagonal fold creates the space to be filled (photographs: MAP).

It is necessary to fold the piece along the line marked by the two diagonals of the lower quarters of the rectangles, as shown below. Since one cathetus of the right-angled triangle that has been formed is double the other, the angle of the fold will be double the tangent of this proportion:



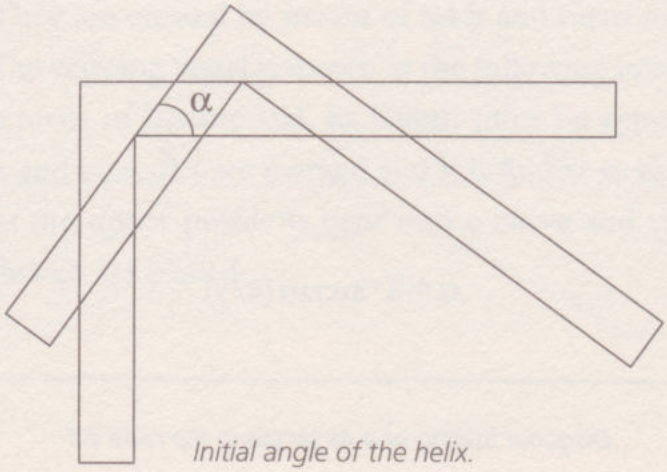
Diagonal folding of a rectangle in the ratio 2:1.

However, this is not the only possibility. According to local customs or the skills of the person, there may be a wide range of receptacles. Furthermore, not everything that is created need be a receptacle. Some objects serve a more ornamental function, such as a helix created by folding thin plant fibres. The four interwoven fibres that make up the following specimen have a width of 3 mm.



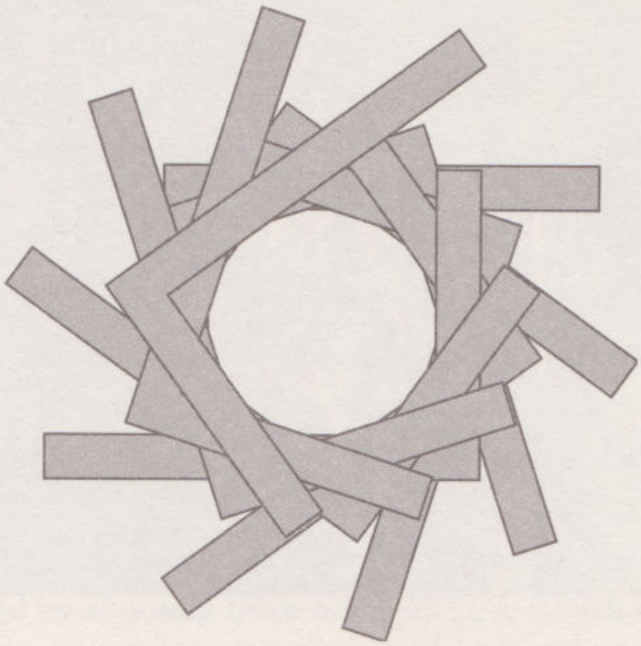
Interwoven helix (photograph: MAP).

The weaving rotates around the axis. The spiral is only supported by the axis at its start and end points. The angles of the vertices of the helix are approximately right angles and have been created by twisting the fibre by half a turn before fixing the fold. These are interwoven in the way shown in the following image. The angle α determines the angle formed by two consecutive vertices, which will be $180° - \alpha$, and the number of sectors for each turn of the helix:



Initial angle of the helix.

As we begin to perform the process, the surface takes shape:

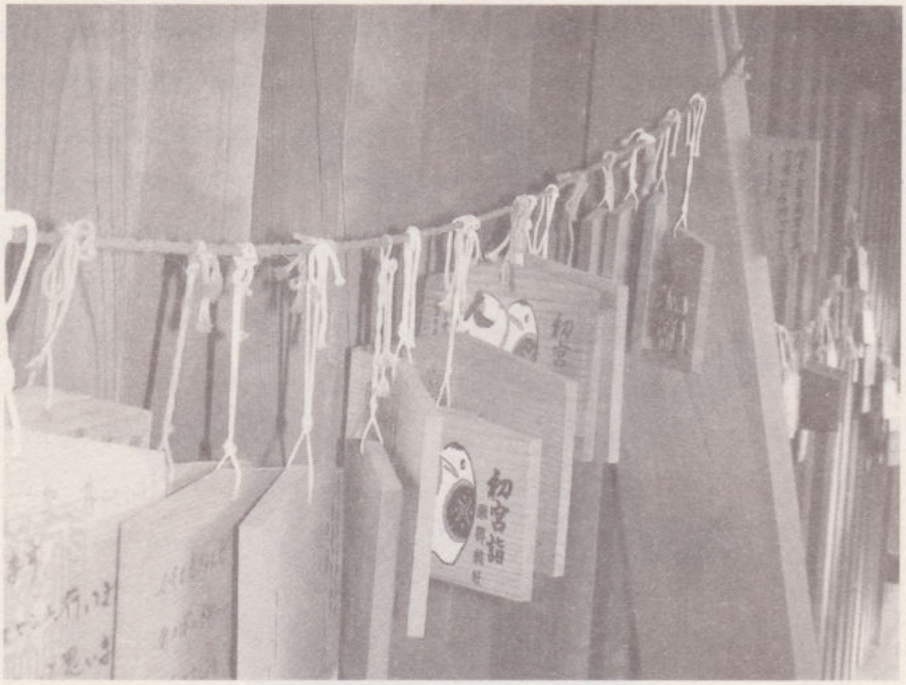


Layout of the interwoven helix.

In Japan, it is traditional for believers to hang up wooden boards at the entrances to shrines asking favours of the gods: students do this before their final exams to obtain good marks. Families and couples do it to ensure they have a good future and businessmen do so in the hope of making easy money.

During the 17th and 18th century, an extraordinary mathematical phenomenon occurred in Japan, whereby large wooden boards were hung from altars with mathematical problems, generally geometric in nature. Some were simple whereas others were extremely difficult. Monks, samurai and other elements of society participated in the challenge of creating and solving these problems. The tablets with the mathematical problems are called *sangaku*. The oldest dates back to 1691 and can be found at the altar of Gion in the city of Kyoto. The last sangaku was a tablet discovered in 2005 at the altar of Ubara in the city of Toyama and dates back to 1879. It has six problems, but there are other sangaku that contain more than 20.

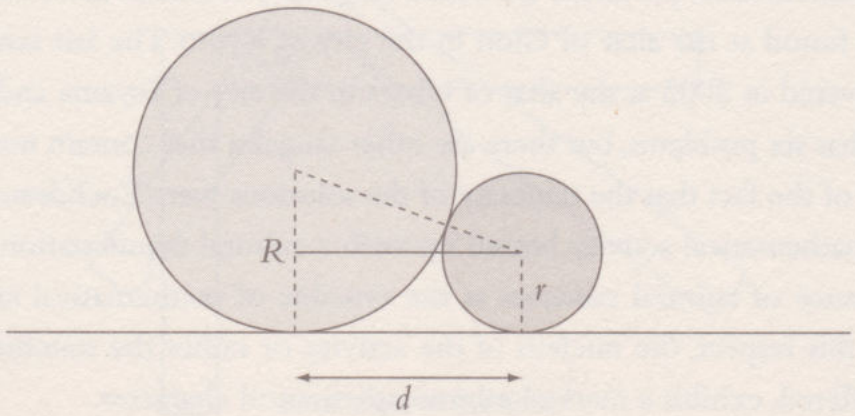
In spite of the fact that the majority of the solutions were Euclidean, this extra-academic mathematical activity bound up with a cultural manifestation underlines the importance of cultural contexts as the crucible of mathematical creation and activity. In this respect, the nucleus of the activity, or rather the statement and the solutions offered, exhibit a marked ethnomathematical character.



Tablets at the entrance to the temple of Hida Kukubun-Ji, in Takayama (photograph: MAP).

The vast majority of problems deal with the inscription of certain geometric shapes within others. Some examples include determining the ratio between the radii of three tangential circles inscribed in another larger circle, finding the dimensions of various squares inscribed in an equilateral triangle, inscribing a series of circles in an ellipse or a series of spheres inside another.

In 1781, Sadasuke published a book entitled the *Essence of Mathematics* and helped his son, Kagen, to prepare the first book of sangaku. It was published in 1789 and is entitled *Mathematical Problems Suspended at the Temple*. A recurring theme in sangaku problems is the tangentiality and inscription of circles in other shapes. Sadasuke's book already includes a simple version of the problem of finding the distance separating the two tangential points of a straight line supporting two tangential circles:



Applying Pythagoras' theorem to the radii R and r of the circles and where d is the required distance, we have:

$$(R-r)^2+d^2=(R+r)^2 \Rightarrow d=2\sqrt{Rr}.$$

The interest extends beyond the problem in its own right and its motivation is related to Pythagorean numbers. Three whole numbers are said to be Pythagorean if they conform to Pythagoras' theorem, or in other words if the square of one is the sum of the squares of the other two. For example, the triples (3,4,5), (6,8,10), (5,12,13) and (119,120,169) are Pythagorean numbers. Three Pythagorean numbers make up a prime Pythagorean triple if the two smallest numbers are relatively prime. This is the case with (3,4,5), (5,12,13) and (119,120,169), but not with (6,8,10), since 6 and 8 are even.

Another problem in Sadasuke's book involves proving that prime Pythagorean triples are obtained by choosing three numbers (a, b, c) such that p and q cannot both be odd where:

$$a=2pq$$
$$b=p^2-q^2$$
$$c=p^2+q^2.$$

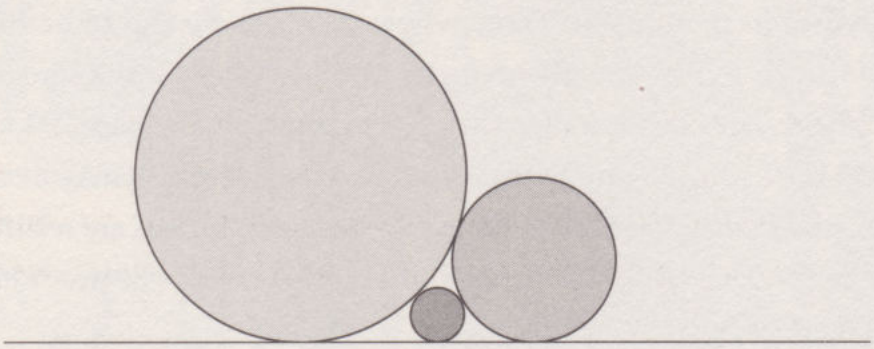
The value of a bears a striking resemblance to the solution to the previous geometric problem. For it to be identical, it would be enough for it to also hold for the square roots of the radii of R and r. Imagine the radii of the circles are squares of whole numbers: $R=p^2$ and $r=q^2$, and that their difference $R-r$ is another whole number s. Hence, the following triple is a prime Pythagorean triple:

$$2pq=d$$
$$p^2-q^2=R-r$$
$$p^2+q^2=R+r.$$

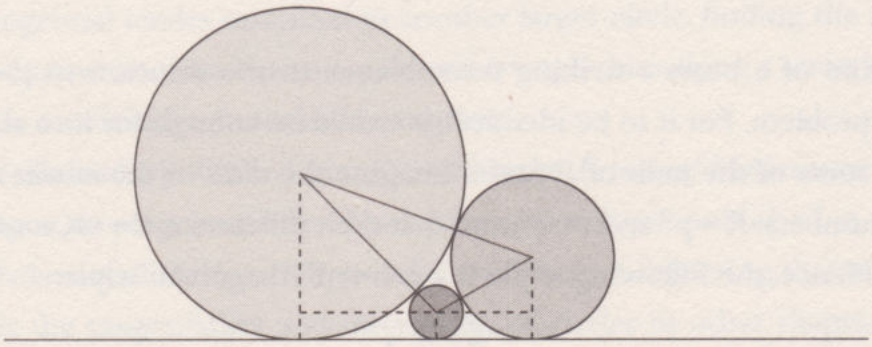
Hence, the algebraic problem is shown to be equivalent to a geometric one. It would appear that this was the vernacular Japanese method for finding prime Pythagorean triples. Finally, another problem asks for all the prime Pythagorean triples for a radius $r \leq 41$. The solutions are:

(3,4,5), (5,12,13), (8,15,17), (7,24,25)
(12,35,37), (20, 21, 29), (9, 40, 41).

Inserting another circle between the two previous circles creates an interesting problem related to a sangaku dating back to 1873, which hangs from the altar of Katayamahiko, in Okayama. What is the ratio between the radii of three circles that are tangential to each other and the line on which they are based?



Once again, Pythagoras' theorem provides the solution. Where $r_1 > r_2 > r_3$ are the radii of the three circles, we can use Pythagoras' theorem to calculate the ratio. To do so, we note the triangle defined by their three centres and the radii joining them to the base:



This gives new right-angled triangles to which Pythagoras' theorem can be applied. Where d_1 and d_2 are the respective bases of the right-angled triangles with hypotenuses $r_1 + r_3$ y $r_2 + r_3$, it can be shown that:

$$(r_1 + r_2)^2 = (r_1 - r_2)^2 + (d_1 + d_2)^2$$
$$(r_1 + r_3)^2 = (r_1 - r_3)^2 + d_1^2$$
$$(r_2 + r_3)^2 = d_2^2 + (r_2 - r_3)^2.$$

Solving for d_1 and d_2 in the second and third equations and substituting the values into the first gives the following expression:

$$\frac{1}{\sqrt{r_3}}=\frac{1}{\sqrt{r_1}}+\frac{1}{\sqrt{r_2}}.$$

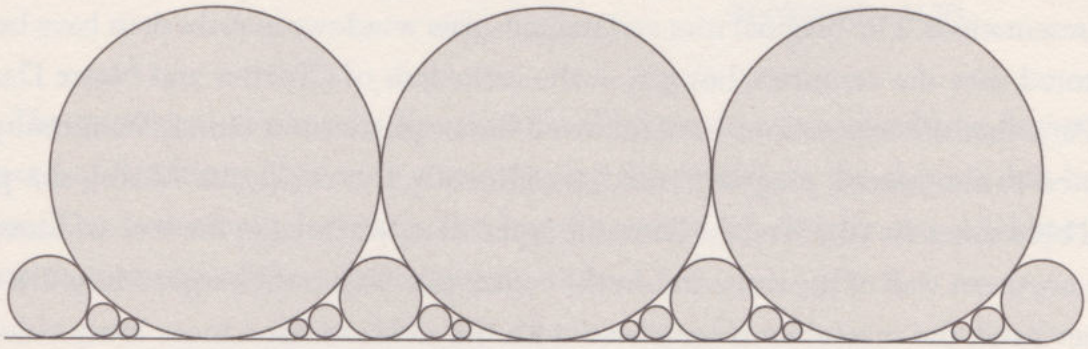
This is the relationship between the radii. This is a dual expression of Pythagoras' theorem, whose character becomes clear by writing the square roots as powers with fraction exponents:

$$r_1^{-\frac{1}{2}}+r_2^{-\frac{1}{2}}=r_3^{-\frac{1}{2}}.$$

How can we find the values of the three radii for which this relationship holds? Is there a triple of whole or rational number solutions? Circles with these properties arise from the inverses of the squares of natural numbers:

$$s_n=\frac{1}{n^2}=\left\{\frac{1}{9},\frac{1}{16},\frac{1}{25},\frac{1}{36},\frac{1}{81},\ldots\right\}.$$

The result is shown below:



Divine rose windows

Tangential circles have not only inspired Japanese monks and samurai. Europe's Gothic cathedrals are also full of figures based on this puzzle. A real-world geometric display of how the circle is used has become a Christian icon. It finds its greatest expression in the rose window, although it is also present in lattice work. The designs are characterised by inserting other circles and series of circles inside a large circle with a diameter of various meters. In the majority of cases, they are tangential to each other and the larger circle. The rose window of the church of Santa Maria del

Pi in Barcelona contains circles in which four tangential circles have been inscribed, which are also tangential to the circle that contains them.

Close up of the rose window of the church of Santa Maria del Pi in Barcelona (photograph: MAP).

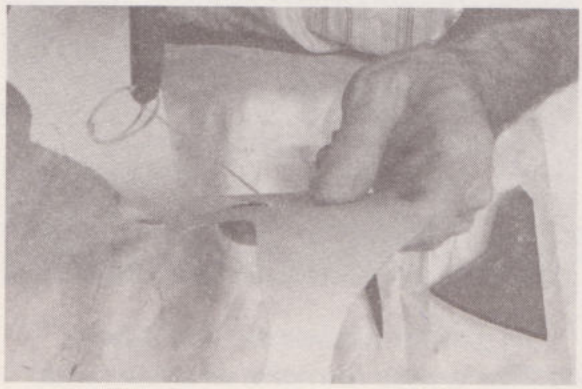
When it comes to rose windows, nothing is left to chance. Everything exists for a reason and that reason is symbolic. Geometry is the basis of the symbolic representations. The original rose and stained-glass windows in cathedrals have been restored over the centuries, but it is in the cathedrals of Chartres and Notre Dame in Paris that the restorations have followed the originals most closely. Femininity is related to the uterus, pregnancy and, traditionally, the night, the Moon, the past and blue tones. At Chartres, the feminine aspect is represented in the rose window of the northern wall of the cathedral, in the centre of which is the Virgin Mary. On the other hand, the masculine is associated with the southern face, more lively colours such as yellow and red, the Sun and the present. The image of Christ is located in the centre of the southern rose window.

Geometry also forms the basis of the symbolic representations of the figures. Geometric similarity, both in terms of shape and proportion, reflects and indicates a relationship between the elements related by them. Nothing is left to chance. Nor is the fact that a rose window is divided into 6, 8, 12, 16 or 24 circular sectors or that it is based on a series of various concentric circles.

In the city of Sabadell near Barcelona there is a workshop that works exclusively with stained-glass windows. The designs are first drawn on paper at a scale of 1:10 before being replicated at the real size. Transfering from the model to finished window used to be done visually, using a pantograph, although new technology is

now used. Using a projector to project the figures, it is possible to magnify a paper design to its actual size.

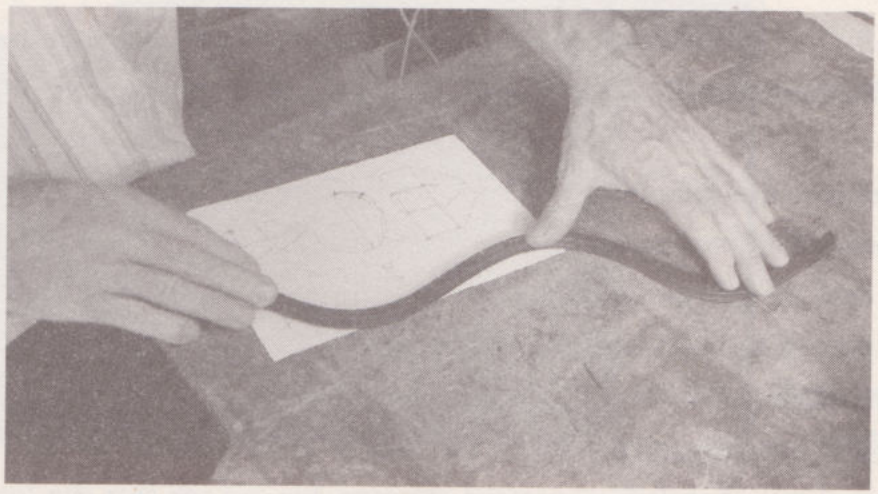
To ensure the stained-glass windows have the right geometric shapes, it is important to bear in mind that a margin of 1.2 mm must always be left between adjacent areas. Instead of always profiling the shapes with this margin using a line drawn at this distance, a triple-bladed set of scissors is used that cuts a margin of this width automatically.



The triple-bladed scissors produce a shaving with a constant width of 1.2 mm.

The two pieces that have been cut are adjusted to the required margin of 1.2 mm (photographs: MAP).

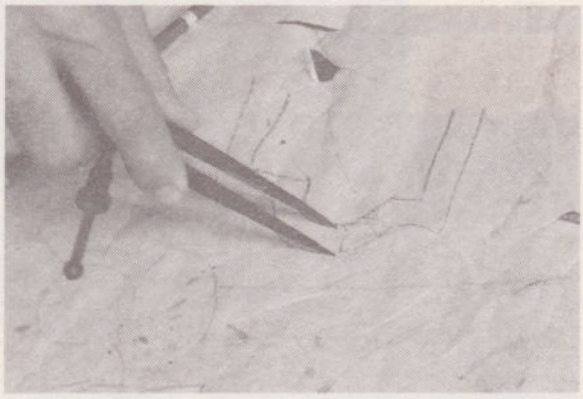
The transferral of curves is also automated by means of a tool called a *flexi curve*, a piece of flexible rubber with a metal spine that preserves the shape it has been given. The instrument facilitates the transformation of circular arcs of shapes in space to segments of the same length on a plane.



The flexi curve conserves the shape it has been given (photograph: MAP).

Another geometric problem that has to be solved by the creators of stained-glass windows is reproducing proportional curves. This problem is solved using a compass, as shown in the images on the following page. Proportionality is guaranteed, provided both curves are intersected by their perpendicular, determining a segment with the same length.

The compass marks the distance between two opposite points of similar curves.



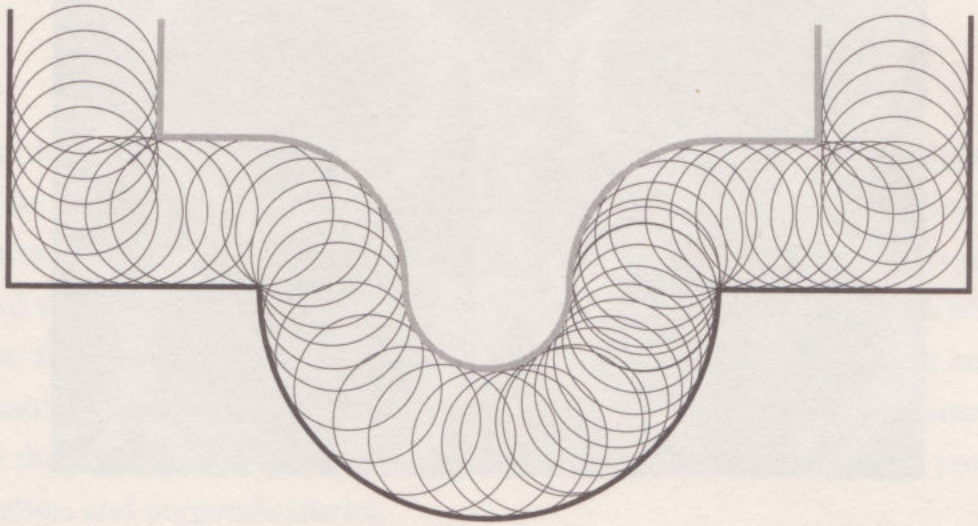
The compass indicates the same distance between another two points on similar curves.

Tracing the profiles of two equidistant curves using a compass (photograph: MAP).

This presents the following problem: are two parallel curves proportional? Are two proportional curves parallel?

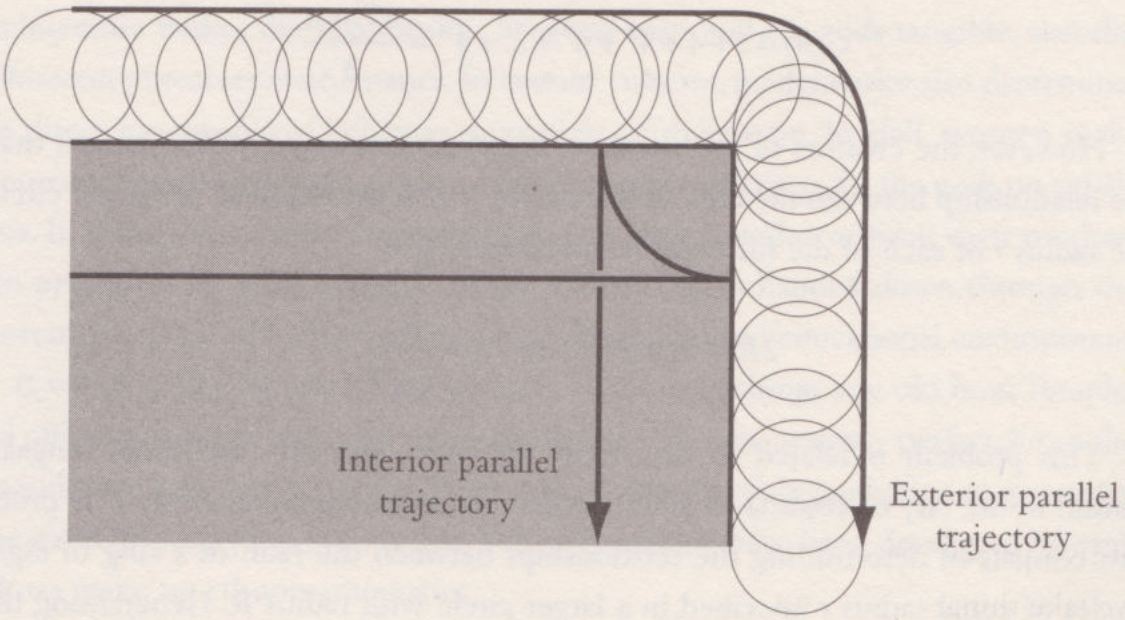
In the case of polygonal curves, there is an equivalence, since all polygonal curves are part of the perimeter of a polygon and the sides of similar polygons are parallel. The same holds for circular arcs. In such cases, the visual image we have of two parallel curves corresponds to the idea of proportional curves. However, if we take things to a slightly exaggerated extreme, we can see that the intuitive idea of parallelism does not correspond to proportionality. For example, the following

two curves are parallel in the sense that the perpendicular at each point is also perpendicular to the other and always determines a line segment of the same length between both. However, the relationship between the two would be different if one was an enlargement or reduction of the other.



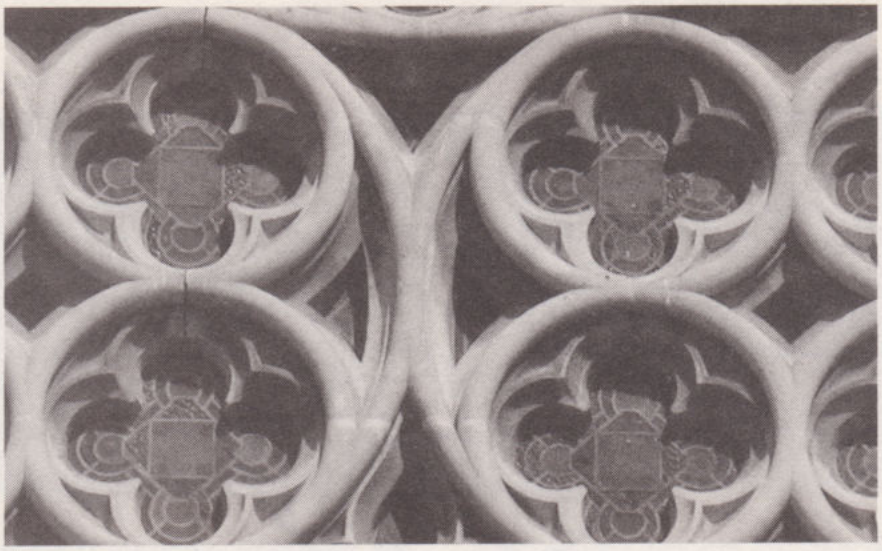
A curve parallel to another curve does not conserve the same angles.

The change in nature between a curve or profile of a polygon and its parallel at a given distance can be clearly seen in the following image. There are two paths or curves that are parallel to the corner of a rectangle: one outside and the other inside. In the case of the outer parallel, the corner disappears, whereas in the case of the inner parallel a loop is formed:



Inner and outer parallels to the corner of a rectangle.

The problem is taken to another level in the latticework of the church of Sant Fèlix, in the city of Sabadell, where each of the four inner circles have been inscribed with other circles. This represents a recurrence of the problem.



Close up of the rose window of the church of Sant Fèlix in Sabadell, Barcelona (photograph: MAP).

Here we can see a circle in which four smaller tangential circles have been inscribed and whose centres constitute a square. Another four circles have been inscribed in each of these following the same pattern. Continuing the same pattern indefinitely, we obtain a series of circles with a quantity $C(n)$ that is obtained by adding powers of 4:

$$C(n) = 1+4+4^2+4^3+\ldots+4^n = \frac{4^{n+1}-1}{3}.$$

However, the creators of the windows were less interested in this pattern than the relationship between the radii of the circles. If R is the radius of the larger circle, the radius r of each of the four circles inscribed in it is:

$$2R = 2r+2r\sqrt{2} \Rightarrow r = \frac{R}{1+\sqrt{2}}.$$

This problem is related to one from the most recently discovered sangaku (found in the city of Toyama in 2005), which we mentioned previously. The problem consists of determining the relationships between the radii of a ring of eight circles of equal radius r inscribed in a larger circle with radius R. Generalising the problem, we can ask what is the relationship when, instead of four or eight circles,

the ring inscribed in the larger circle is made up of n circles. The solution is provided by means of a trigonometric analysis of the problem:

$$r = \frac{R}{1 + \dfrac{1}{\sin\left(\dfrac{\pi}{n}\right)}}.$$

As we have already seen, the presence of architectures from outside Europe in the Middle Ages illustrates mathematical thought beyond Europe at this time. Such architecture, like the architecture in Europe, is based on the dialogue between the circle and the square, and there would be no religious architecture at all without these two fundamental geometric shapes. Just like the Egyptian pyramids and the Babylonian ziggurats, temples and tombs throughout the world have been constructed based on these two geometric shapes. Implicit in their construction are the concepts of parallelism and perpendicularity.

Increasing the dimensions of these shapes leads to various three-dimensional versions. On the one hand are the hemispheres of the Buddhist stupas in India and Nepal, which culminate with a 'die' (cube). On the other, consider the stepped pyramids of pre-Colombian America. The Islamic mosques of the Middle East gave a further dimension to the circle as a curve, transforming it into a helix, or spiral, ascending to the heavens.

The way in which the beliefs of a culture are expressed is a fundamental aspect. Architecture makes the relationship between man and his gods tangible, and this architecture involves mathematics. In certain cultures, mathematics also determines the direct expression of believers, regardless of their type. In Bali, women make geometric vessels and envelopes in which they place offerings for the gods on a daily basis. In doing so, they put mathematical ideas they have learnt from their mothers into practice. This is an example of knowledge that is handed down through the generations and is not linked to the formal academic and educational environment.

If we are going to respect the gods, we cannot do things any old how. Temples and offerings must be executed correctly and be – in some senses – perfect. From the examples we have seen so far, it appears that all cultures relate perfection to geometry. The mathematical ideas required for this purpose that have been developed by each culture make up ethnomathematics.

Chapter 4

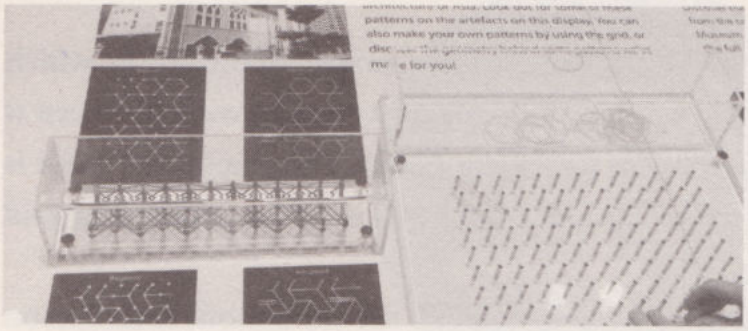
Beauty is More Beautiful When Geometric

It is not true that geometry increases the beauty of things. The title of the chapter alludes to the value all cultures have placed on doing things correctly. They deserve to be described as such thanks to the rigour with which they have been carried out. And when it comes to shapes, rigour is closely related to geometry. This forms the basis of Ernst Gombrich's discussion of art in his book *The Sense of Order* on ornamental art.

Go geometric

In just a few years, airports throughout the world have turned from waiting areas for flights into shopping centres with aircraft. There is a wide range of options on offer: kiosks, pharmacies, bars, restaurants, jewellers, clothes stores, gift shops, electronics stores... Passengers can purchase all sorts of products as they wait for their flight to take off or move from one terminal to another.

However, there is another type of offer available. Some airports, such as Changi in Singapore, offer passengers free cultural activities. Passengers in recent years have been able to enjoy one such activity passing through the island state. In one of the vestibules, there were various panels with an exhibition entitled *Go Geometric*. On the one hand, the exhibition highlighted the relationship between culture and geometry; on the other it invited visitors to try out activities in which they could recreate some of the geometric patterns on display, present in the architecture and crafts of various cultures in Asia.



Go Geometric *at Changi airport in Singapore (photograph: MAP).*

One of the activities consisted of printing a seal with a special symbolic design on a piece of paper. The *endless knot* is one of the emblems of Buddhism. It is a linear shape the path of which can be traced in one stroke. Its name was inspired by this topological property. It is often used as an ornamental pattern for various objects, such as the simple version that decorates the profile of the plate shown in the following image:

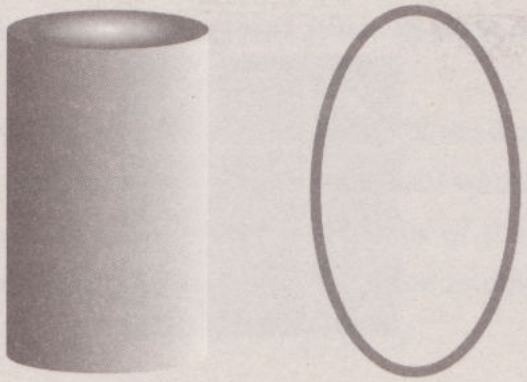
The activity for the endless knot of Buddha at Changi airport and a printout of the knot (photographs: MAP).

What makes the knot endless? Clearly the cyclical character of the line by which it is defined. If we follow the line starting from any of its points, we will return it to the origin after passing through the remaining points. The line of the knot is continuous and closed. It all depends on the configuration of the underlying grid and the way the knot is woven into it.

Two shapes are said to be topologically equivalent if the continuous deformation of one (without cuts) gives rise to the other and does not change the number of holes. Hence, a ring and a picture frame are topologically equivalent. The same holds

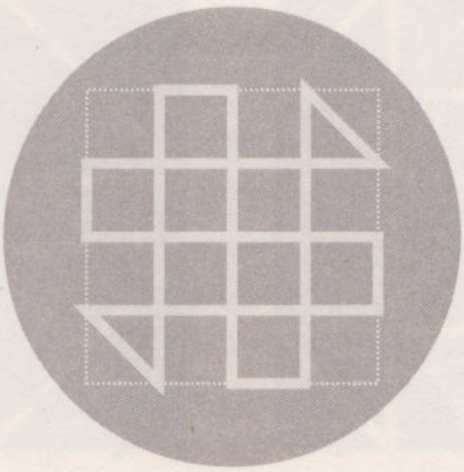
TOPOLOGY

Topology is a branch of mathematics that studies shapes without paying attention to their measurements, be they of lengths, angles, surfaces or volumes. From the perspective of topology, all objects are soft and deformable. If, by means of a continuous deformation, or in other words, without rips or breaks, two objects can be transformed into the same shape, they are topologically equivalent. For example, all polygons are topologically equal and are equivalent to the circle. The same can be said of polyhedra and the sphere. A t-shirt and a piece of paper with four holes are also topologically equivalent, the number of holes serving as a topological indicator. A ring is topologically equivalent to a cup, since it has the same number of holes. This is not the case with a glass, which does not have any holes. In contrast, a spoon, knife and a fork are equal since they do not have any holes at all.

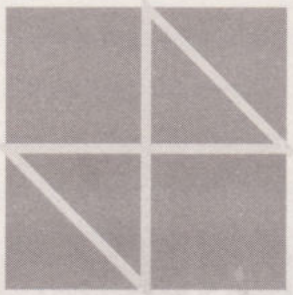


A cylinder and a ring are topologically equal.

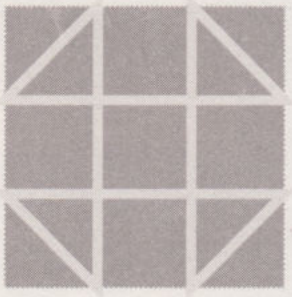
for the endless knot in the previous image and the following design. Furthermore, both have symmetry of order two (180°):



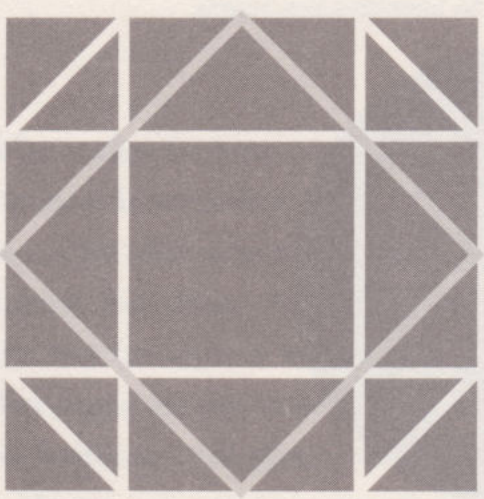
The cyclical itinerary with rotational symmetry of order two is achieved using three vertices of the grid on each side of the square. The same holds if there is just one:

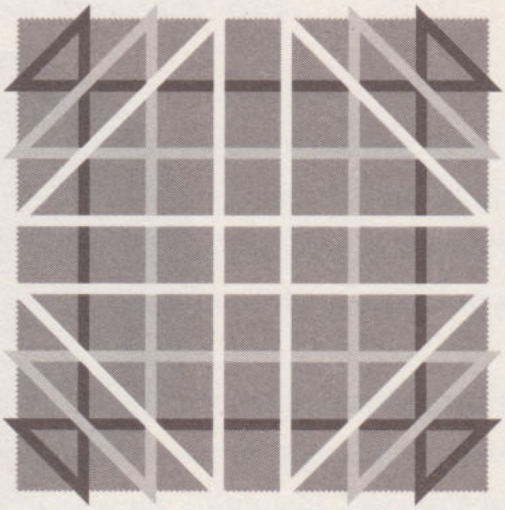


If the number of vertices on the grid of each side is even, we have a different type of circle, one with a rotational symmetry of order four (90°):

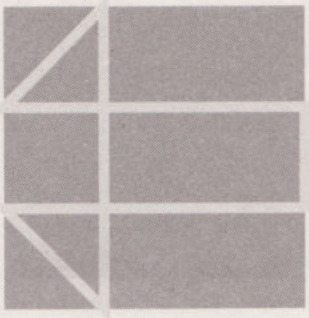


With the exception of the case of a single vertex on one side, it is possible to construct various cycles of this type (with order-four rotational symmetry) with any number of vertices on each side, even or odd. Two, in the case of a grid with 4×4 cells, and three, for a 7×7 grid:





When the number of vertices of the grid on the side is even (the grid has an odd number of cells), there is no cycle that passes through all the vertices and it will be of the original type:



Hence, to create an endless knot of the original type that passes through all the vertices of the grid, it is necessary to have an odd number of vertices on each side of the grid, which is the same as saying that it has an even number of cells:

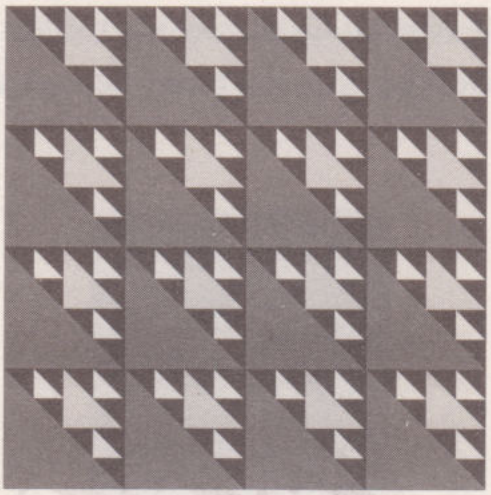
Theorem 1: If the grid has an even number of cells, the knot is endless and of the original type with order-two rotational symmetry (180°).

Theorem 2: Regardless of the number n^2 of cells of the grid, with $n = 2 \cdot k$ or $n = 2 \cdot k + 1$, it is possible to create k cycles with order-four rotational symmetry.

We have previously seen that in a grid with 49 cells, $n = 7 = 2 \cdot 3 + 1$, there are 3 cycles with symmetry of order four. In one with 16, $16 = (2 \cdot 2)^2$, there are two.

Variations on a theme: symmetry

Geometric designs are universal. It is extremely rare to encounter cultures that have not created geometric designs for systematic use as ornamental emblems, symbols or patterns. This has been the case since remote times, dating back to the geometric petroglyph in the cave of Blombos. Those of ancient Egypt, classical Greece and the Byzantine world are more formal, despite still being before our era. In our era, the Roman world used geometric patterns in mosaics, reaching their greatest splendour in Venice prior to the Renaissance. During this period we can find a Roman-Byzantine design, purely geometric, containing a recurring pattern of a fractal nature.



Roman-Byzantine design (circa 700 AD).

A square has been divided into 16 cells. The diagonal divides each cell into two right-angled isosceles triangles. One of these has been shaded grey, and the other has been divided into another four triangles similar to the previous one. One of these has been shaded light grey, and the other three have been divided into another four right-angle isosceles triangles. Each of these three triangles is surrounded by another three, which gives $3 \cdot 3 \cdot 16 = 9 \cdot 16 = 144$ new triangles. From here, the process could continue indefinitely. For each new stage, the number of triangles in the previous stage is multiplied by three. Starting with the yellow triangles:

Stage	New triangles	Total triangles
1	16	16
2	$3 \cdot 16 = 48$	64
3	$3^2 \cdot 16 = 144$	208
4	$3^3 \cdot 16 = 432$	640
5	$3^4 \cdot 16 = 1{,}296$	1,936
...	...	...
N	$16 \cdot 3^{N-1}$ $(N>2)$	$24 \cdot (3^{N-1}-1)$

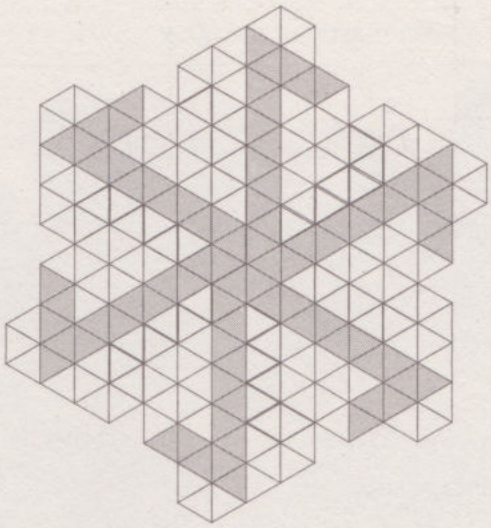
The design has *cm* reflective symmetry determined by the parallel axes of symmetry of the ascending diagonals in each cell.

However, there is a culture that has taken geometric design to extraordinary heights, to such an extent that when it comes to geometric design, it would be possible to say that it has all been done before. This is Islamic culture. The presence

of Arabic designs and mosaics extends from Morocco to India, and from Spain to Zanzibar (Tanzania) and praise for the symmetry of Arabic ornamentation has become common, not just in mosques, palaces and madrasas, but also in hotels, airports and planes. The roots of Islamic designs date back to Arabic designs that originate before the year 1000.

Arabic design (around 1200 AD).

This Arabic design consists of the repetition of a hexagonal shape with 60° rotational symmetry. Its development produces a tessellation or tiling of the entire plane. It is based on a grid of equilateral triangles that are combined to create the fundamental shape or *leitmotiv* of the design.



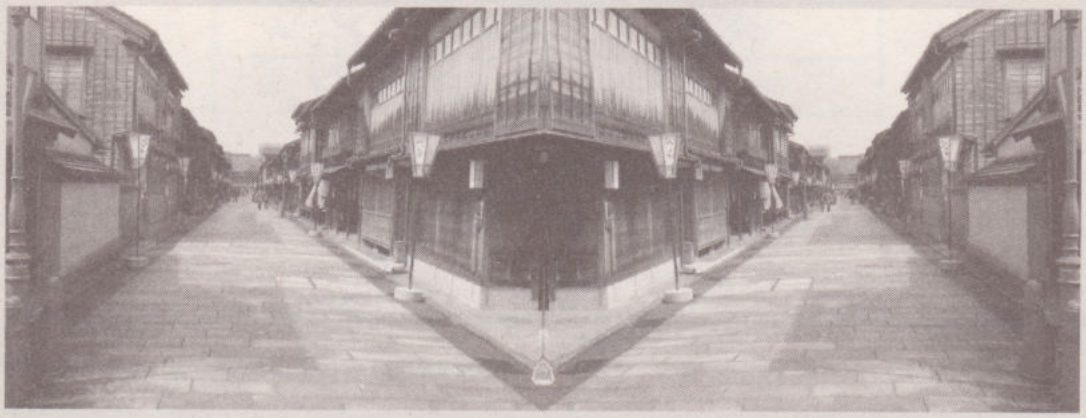
Many stand out on account of the use of triangular grids instead of rectangular ones, meaning that the typical angles used in this type of ornamentation are 60° and 120°.

The right angle is also present, although it is not dominant. In an Islamic context, geometry becomes more sophisticated with the appearance of double lines in the form of ribbons that interweave and form knots. The design is two-dimensional, although it plays with the perceptions of the spectator, creating a three-dimensional effect. The equilateral triangles of the grid combine to create an endless number of compound shapes, which include the six- and twelve-pointed stars, such as the ones in the Alhambra in Granada.

Nazrid design of the Alhambra in Granada (Spain, ninth century).

SYMMETRY AND IMPOSSIBLE WORLDS

We know that many of the streets down which we walk on a daily basis are straight parallel lines. However, it is not surprising to see their ends converge on a distant point of confusion on the horizon. Our vision and the straight-line trajectory of the light reflected on objects means they appear smaller the further away they are. Symmetry and technology can be combined to create impossible worlds, albeit based on reality. It is enough to take a photograph, apply a vertical or horizontal reflection, and join it to the original. The double image shows two parallel streets, one a symmetrical transformation of the other.



A street in Kanazawa (Japan) and its symmetry (photograph: MAP).

Unfortunately, not much is known about how the mosaics of the Alhambra were executed, nor how regular polygons with nine sides were drawn – whose perfect construction using a ruler and compass Gauss proved to be impossible in the 18th century. When it comes to the procedures for carrying out these designs, at best we can speculate and guess. However, we shall see below that there are still cultures whose designs are carried out on a daily basis using the same techniques as before.

Indian kolams

Every morning, women in the south of India, especially in the states of Tamil Nadu and Kerala, carry out a ritual outside the doors of their houses that consists of drawing a series of shapes on the ground by hand. The lines are drawn using rice powder or chalk and the final shapes must be white or coloured with vibrant colours. They are called *kolam* and may range from small and simple floral representations all the way to enormous and complex geometric designs.

The kolam are works of art, but not just art. The lines and shapes from which they are composed are often based on a straight-line structure whose points are previously marked on the ground. This structure acts as a grid for the design that is to be drawn. Furthermore, it is made up of smaller figures, generally symmetric, which are repeated by following a specific pattern that is also determined by the straight-line shape of the points. The following photograph shows a kolam with two perpendicular axes of symmetry based on a grid of octagonal points.

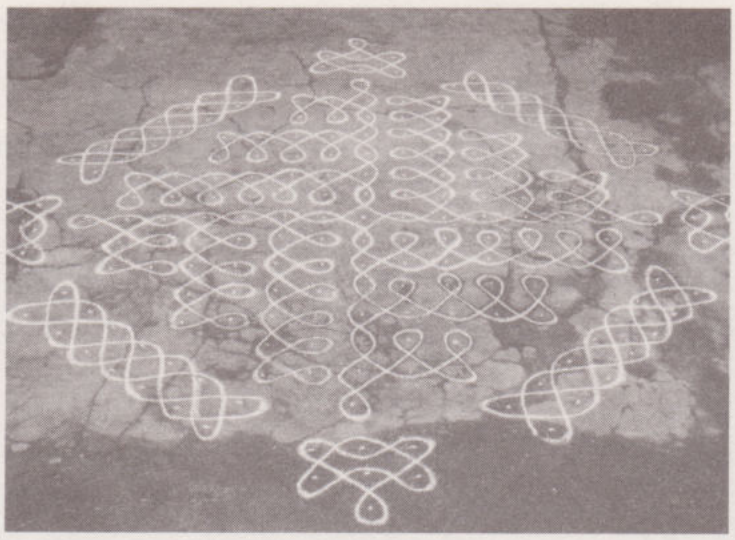
Execution of a kolam in Chennai, Tamil Nadu, India (photograph: Kamini Dandapani).

The fact that the kolam are executed by women is not a result of established rules, but the consequence of a custom that means they are responsible for the task, like other domestic duties. Men are not prohibited from drawing kolam and some men may execute them simply for the enjoyment of doing so.

One extraordinary case in which the execution of the kolam is usually performed by a man is a special ritual dedicated to the mother goddess Bhagavathi, in Kerala. This ritual, called Bhagavathi Sevai, can only be carried out by a priest, who is always a man, and who must thus draw the kolam. This is specifically referred to as a *padman* (lotus).

There are two basic types of kolam. Some are similar to the image on the previous page and are made up of two-dimensional shapes that fill the spaces created by a grid of points. Others consist of one or more continuous lines that pass through all the points of a grid, interwoven into one or more shapes.

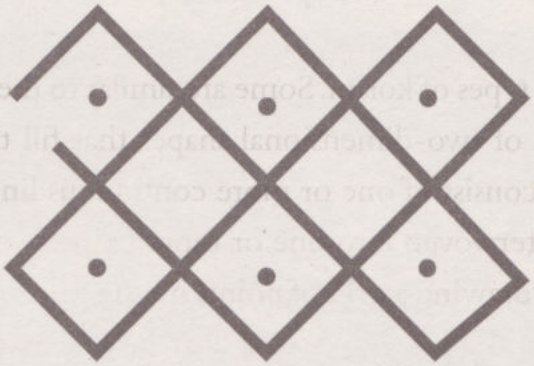
All kolam begin by drawing a grid of points on the ground. Their layout depends on the available space, and the configurations can be practised on paper beforehand, especially in the case of extremely complex or large shapes. The drawing of the lines going around the points must be done confidently. It is frowned upon to make errors that require the kolam to be corrected. The shapes do not have a specific name; instead the names of similar shapes are used to refer to them, such as stars, lotus flowers, palm trees, temple chariots, etc. The interweaving lines are figures of eight or endless knots amplified to reach degrees of great complexity such as the one shown below:



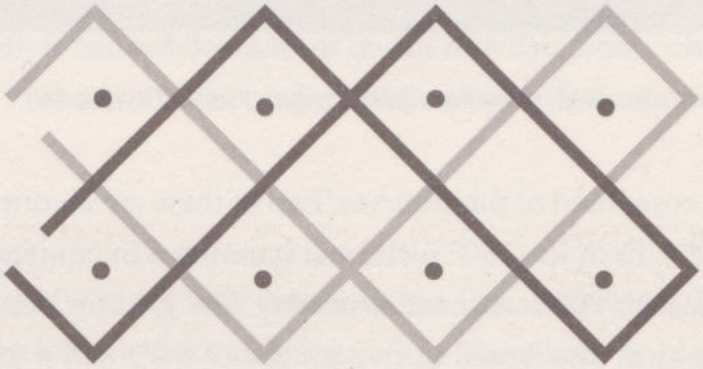
A kolam with different elements composed from a single line (photograph: Kamini Dandapani).

The similarities with the symbol for infinity is not a coincidence, since in this region continuous lines of this nature are a symbol of the never-ending cycle of life, birth, fertility and death.

Carefully studying the curved sides of the previous kolam we can see what can be done with a single line. The four side shapes are rectangular and based on two grids of 2×7 points. A single line covers the whole grid, passing through all the points. The same can be done with 2×3 and 2×5 grids:



However, not with a 2×4 grid. In this case, two lines are required, both of which are vertically and horizontally symmetrical to each other:

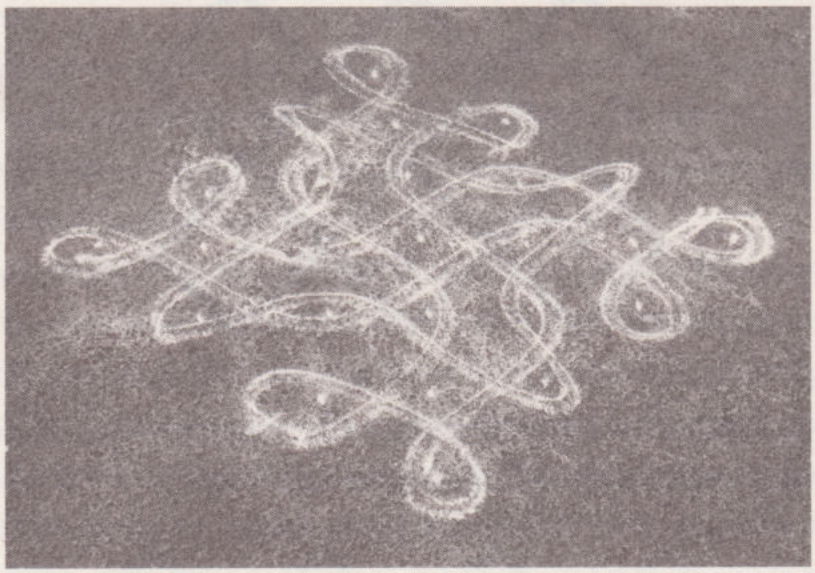


The possibility or impossibility of covering these grids with a single line is determined by the odd or even nature of the number of columns. Effectively, numbering the columns from left to right, the series of columns through which the curve passes in each of the lines passing through the 2×3, 2×5 and 2×7 grids are, respectively: $\{1, 2, 3\}$, $\{1, 2, 4, 5\}$ and $\{1, 2, 4, 6, 7\}$. This type of pattern is not possible if the number of columns is even.

Given a grid of points with two rows, A and B, and N columns (for N odd: $N = 2 \cdot k + 1$), the pattern that must be followed to draw the curve connecting them all with a single line is:

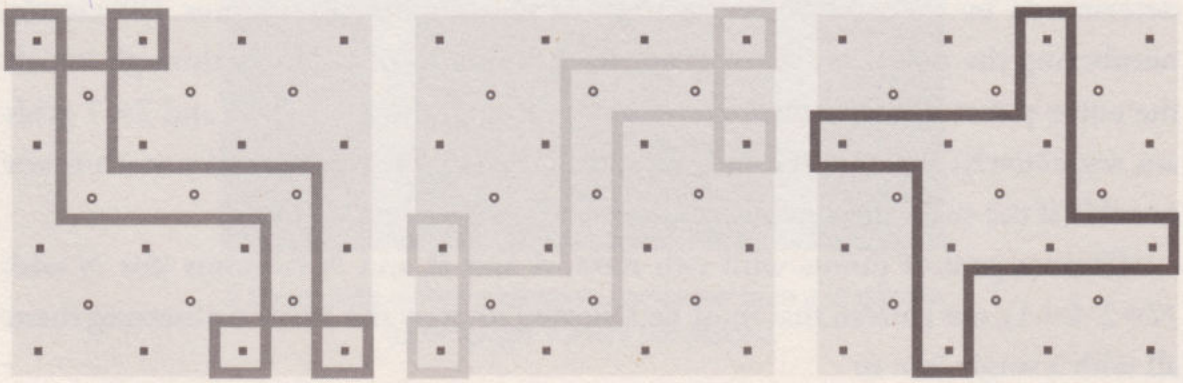
$$N = 2 \cdot k + 1:$$
$$k \text{ even: } \{A(1), B(2), A(4), B(6), \ldots, A(2 \cdot k), B(N)\};$$
$$k \text{ odd: } \{A(1), B(2), A(4), B(6), \ldots, A(2 \cdot k), A(N)\}.$$

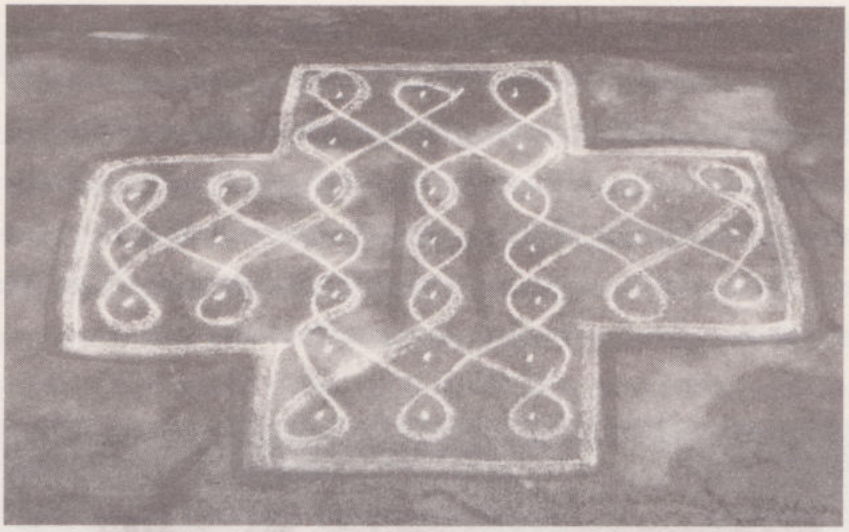
Some kolam consist of a single curve, following the ideal of the endless knot mentioned at the start of this chapter, but the majority are composed of various curves, such as the following:



Kolam with three lines (photograph: Kamini Dandapani).

This kolam is composed of three curves. Two of these are identical and are related by a rotation of 90°. Each has 180° rotational symmetry. In contrast, the third curve makes up a shape with 90° rotational symmetry. The grid for the design is double. It consists of 25 points distributed in two groups of 3×3 and 4×4, the first inside the second:





Kolam (photograph: Kamini Dandapani).

The drawing of kolams in the south of India dates back many centuries. Their origins may be related to similar shapes that were drawn in Central Africa. If they exhibit evidence of any mathematical thought it is seen more in the final symmetry of the shapes than the methodological rigour with which they are drawn. In this context, the women are the bearers of a centuries-old tradition and a mathematical knowledge that they display in front of their houses on a daily basis. The method for drawing the kolam is handed down from mothers to daughters, who have developed and extended it to the limits, fascinating mathematicians throughout the world.

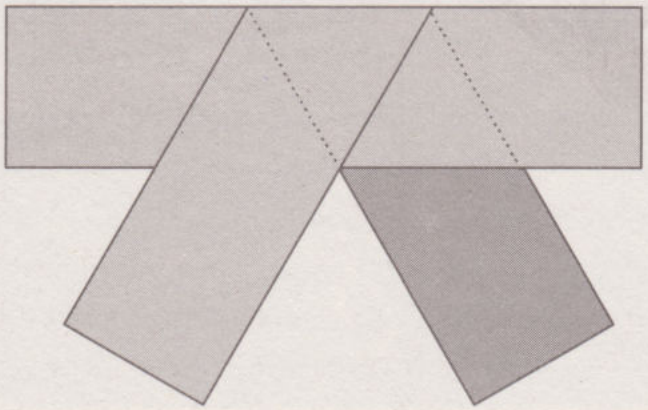
Weaving

The 11th chapter of *Tao te Ching* (attributed to Lao-Tsu) states that the usefulness of wheels, vessels and windows originates from their hole. It is true that man has sought to enclose small portions of the infinite space that surrounds us since prehistoric times. However, the problem of delimiting space dates back to a previous problem, since it is necessary to create a frame with which to enclose it. The wheel requires a circumference; the urn, a spherical surface; a window, the plane of the wall in which the hole is made.

Throughout the ages, flat and curved spaces have been created using an infinite range of materials and techniques. Perhaps the weaving of plant fibres to create surfaces and volumes is one of the most universal activities. Interlacing various lines, such as the fragments of a branch, it becomes possible to create matting, walls and roofs. These are surfaces, but with these same elements, humankind has been able

to weave volumes, such as baskets, cages for crickets and cockerels, and *takraw* balls, which are used in a Southeast Asian ball game played with one's feet.

The creativity and skill of craftsmen throughout the world in basketmaking has brought praise in terms of both technology and art. The activity uses many mathematical ideas. Paulus Gerdes, an ethnomathematical researcher from Mozambique, has studied the patterns and shapes deriving from this craft. The geometric problems related to basketmaking include the following: one fibre must go around another of the same size. What is the angle of the fold to which this gives rise? The answer of 60° is obtained using a trigonometric calculation. In practice, the angle is obtained by folding the ribbon as shown below:



Ornamental geometric creativity in basketmaking (photograph: MAP).

Takraw balls

Takraw balls are woven in Southeast Asia using pieces of rattan, a palm cane that is also used to make furniture. It is similar to wicker, although it is not round but flat. In spite of being flexible, it is extremely hard and does not break easily, even when kicked, as in the case of the game of takraw.

A takraw ball and its maker (photograph: MAP).

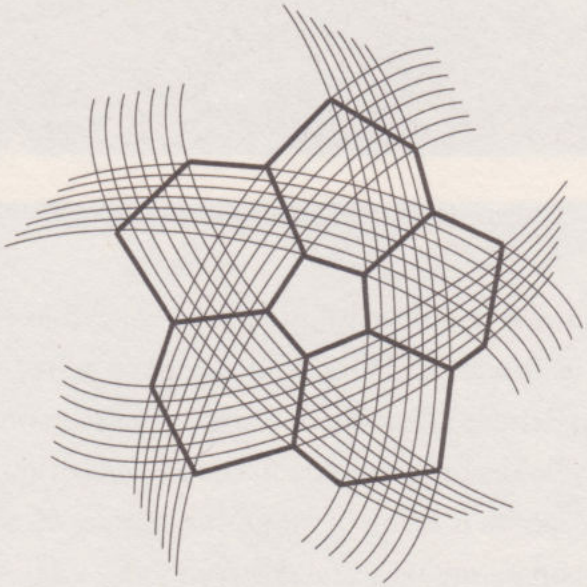
The person making it did not follow a system, design or make calculations. Observing the process by which it is made, it is hard to believe it is possible to make such a perfect sphere without using mathematics. However, mathematics is not only verbal or written, but also exists implicitly in the minds of those who think about it.

In mathematics, a sphere is made up of points identically spaced from one other point, referred to as the centre. However, the weaving of this takraw ball was not done in this way. The craftsman followed a precise and effective method to achieve a perfect ball (let's ignore the inherent imperfections imposed by reality). The essence of the ball is not composed of a centre and a radius, but is instead made based on a polyhedron with a constant curvature. The work begins by interweaving five strips of rattan to form a pentagon that is as regular as possible. Next, selecting some of the ends, the craftsman introduces another strip into the process. One of its ends has already been tied onto a circumference that will determine the diameter of the ball.

Following the features of the strips, the pentagonal faces initially appear as gaps in the warp. They will be filled with the remaining ends of each strip as the work progresses. Essentially, the object is similar to the one produced by cutting the vertices of an icosahedron, which are pyramids with a pentagonal base. By cutting them at half their height, we create 20 pentagonal openings which, being covered by one face, give a semi-regular polyhedron. This is called a truncated icosahedron. This is the geometric shape that craftsman are really weaving, a truncated icosahedron with 60 vertices, 90 edges and 32 faces, 20 of which are hexagonal and 12 of which are pentagonal. The rattan tends to recover its tension naturally, giving the ball its constant curvature. The triple intersections of the six resulting strips of fibres, each originating from one of the fibres used, determine the 20 hexagonal faces of the ball:

$$\binom{6}{3} = \frac{6!}{3! \cdot (6-3)!} = 20.$$

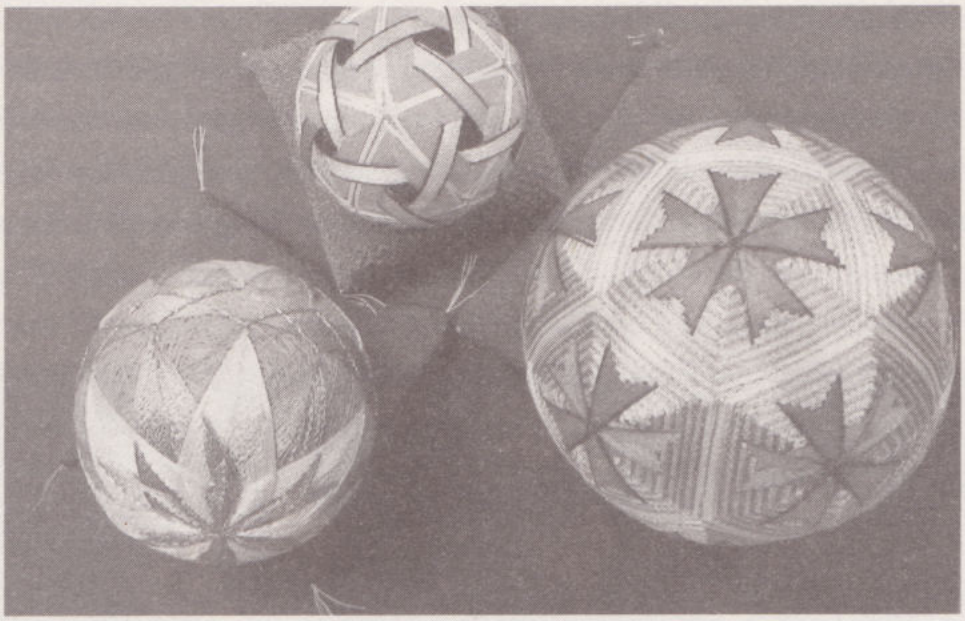
See for yourself:



Temari balls

Japanese *temari* balls have their origins in China. Initially they were made using deerskin and their use was restricted to the court nobles who used them for sporting entertainment. When the ladies of the court began to knit balls with silk threads, they took on a new role as decorative elements. Competitions were run to find the most elaborate termari ball in terms of the pattern and colour.

The art form dates back to around the year 1000 and has been handed down from mother to daughter throughout the generations. Its popularity increased with the passing of time and new techniques for making the balls were developed. However, the invention of rubber balls resulted in the interest in temari declining. Today however, this traditional art form is once again highly prized and has become sophisticated to the point that there are temari ball associations in Japan.



Japanese temari balls (photograph: MAP).

The nucleus of a modern temari ball is another ball made of expanded polystyrene or plastic. It is best if this inner ball is soft, allowing pins to be stuck into it. The majority of weaving designs are geometric and are extraordinarily rigorous in their execution, giving the object its final characteristic appearance.

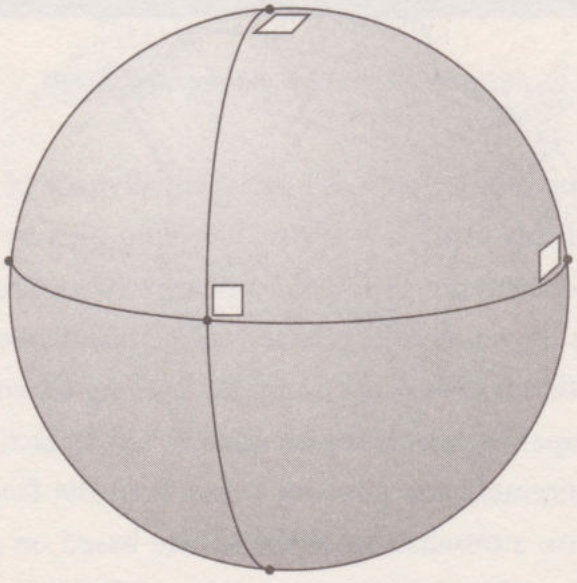
One of the tools that is extremely useful for making temari balls is a 'V'-shaped forked ruler with an opening that is approximately 72°. In fact, it is two rulers joined at the ends. The requirement for this tool stems from the fact that a large number of temaris are based on a tessellation of the sphere based on a dodecahedron. This means working with regular pentagons and a system of five radial axes. Dividing the 360° of a full-turn into five parts gives the 72° of this ruler.

One of the first tasks that must be carried out in making a temari is to divide its surface into eight equal parts. Doing so involves making use of a fundamental difference between the plane and a curved surface such as the sphere. On the plane,

the sum of the angles of a triangle is always 180°. On the sphere, and the surface of the temari is divided in this way, the sum of the angles of a triangle can be 270°.

The process is as follows. First of all, a pin is used to mark an arbitrary point on the ball. Based on this point, a strip is wrapped around the ball, such that the turn also passes through the pin. The point is marked on the strip, which is cut at the mark, making it the same length as the perimeter of the ball. Both ends of the strip are then folded together, and the fold is marked. The action is repeated for each of the two halves. In this way, the strip will have marks corresponding to its quarter, half and three-quarter parts.

The strip is now hooked onto the pin and is wrapped around it. A new pin is placed on the midpoint. The previous will be the north pole and this will be the south pole. Turning the ball over and arranging the strip so that it is perpendicular to the axis of the two poles, the midpoints are marked using two pins. Six pins will now be inserted into the ball; these will be the vertices of six equilateral spherical triangles. However, the angles of these spherical triangles will not be 60° but 90°. On the surface of a sphere, equilateral triangles have right angles. The three perpendicular axes (meridian circles) determined by the six pins divide the surface of the sphere into eight equal parts.



These axes can form the basis of a simple design using coloured threads. Using them to mark more meridians or parallels we can divide the sphere into more spindles, such as the temari balls on the left of the photograph on page 115. Their design

emphasises the equator of the ball, while meridians begin at its poles such that they create 24 spindles (12 of each colour) each at 15°. The other two balls in the image have the pentagonal structure of the dodecahedron.

In fact, any Platonic solid, not just the dodecahedron, can form the basis of a temari ball.

GAMELAN MUSIC

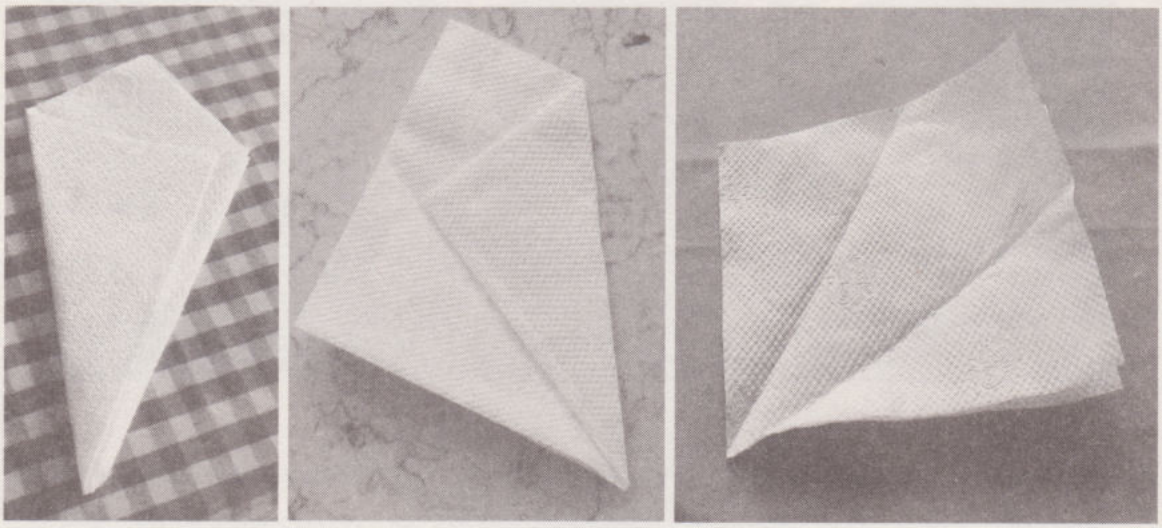
Gamelan orchestras are indigenous to Java and Bali and are composed of a series of one or more large gongs, a pair of drums, at least four pairs of cymbals, a pair of small groups of gongs of between 8 and 14 units each, and flutes. The most typical section of the gamelan are its metal xylophones of different sizes, varying from 7 and 12 notes, which are struck using special hammers.

The musical compositions are divided into cyclical sections that follow patterns based on the powers of 2 and consist of tempos of 2, 4, 8, 16 or 32. This determines the speed the music is played. The melodic ornamentations are played 4 or 8 times faster than the melody, and this, in turn, makes it 2, 4 or 8 times faster than the rhythm section. The duplication makes it easier to keep rhythm and endows the music with its dynamic nature.

Napkins and origami

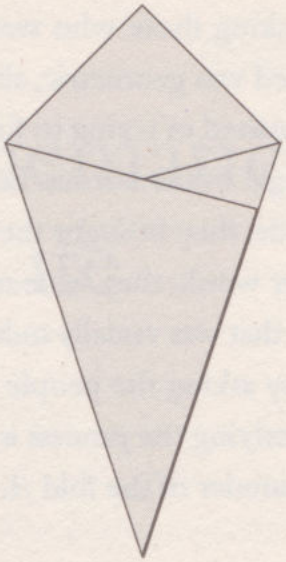
There is a way of folding paper napkins that can be seen in restaurants all across the world's largest archipelago, Indonesia. There, in any Indonesian *warung*, or cafe, napkins are folded in a specific way. From Sumatra in the west to Irian Jaya in the east, the waiters know how to fold a napkin in the Indonesian style.

The table of an Indonesian warung *(photograph: MAP).*



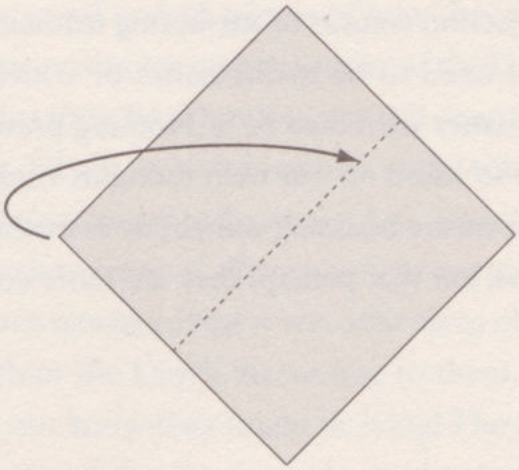
Unfolding napkins taken from different restaurants.

This style consists of folding a square napkin in such a way that the folds divide the right angle of one of its vertices into three equal parts. In this way, a symmetric quadrilateral is created with a right angle, another at 30° and two more angles of 120°.



Napkin folded in the Indonesian style.

For a long time, I believed the necessary reference point for making a good fold would have to be the one I would have taken, which consisted of bringing the corner of the napkin to the midpoint of one of its adjacent sides:



Folding the napkin by bringing one of the side vertices over the bisector while keeping the lower corner fixed.

This gives a right-angle triangle with a cathetus with a length that is half the hypotenuse and, hence, with an angle of 30°. When I had the opportunity to see how certain waiters carried out this task, it appeared to confirm this model, since they clearly directed one corner of the paper to the midpoint of the opposite side of the napkin.

However, I was mistaken. Asking those who were doing the folding, I was told that the reference point they used was geometric, although it was not the one I had imagined and observed, but consisted of trying to fold the napkin such that the side marked the third part of the angle of the corner. Instead of bringing the corner to the midpoint of the opposite side, they brought the side towards the halfway point of the unfolded section. In other words, they were trying to find the bisector of the rest of the fold. It was a solution that was visually indistinguishable from my own and that could only be discovered by asking the people who were folding the napkins.

The mathematical idea underlying the process is that $3 = 1 + 2$. Letting R be the angle corresponding to the remainder of the fold A, which has been made, we have:

$$90° = R + 2 \cdot A.$$

And since we want the folded angle to coincide with the angle of the remainder, the result is the trisection of the right angle of the corner of the napkin:

$$\left.\begin{aligned} 90° &= R + 2 \cdot A \\ A &= R \end{aligned}\right\} \Rightarrow 90° = 3 \cdot A \quad \left(A = 30°\right).$$

A mathematical projection consists of attributing mathematics to a phenomenon whose reality does not need to be mathematics or which, being mathematical, does not fit the mathematics attributed to it. Nothing prevents us from projecting mathematics, but to do so based on our own thoughts runs a risk. It is not that by means of our projection we are branding somebody as mathematically incompetent when this is not the case, but that perhaps they are more competent than you.

Chapter 5

Ethnomathematics in Everyday Life

Popular logic

The Dayak (Borneo)

Alfred Russel Wallace was a British naturalist who travelled around the Malay archipelago in the second half of the 19th century. A contemporary of Charles Darwin, he studied the flora and fauna of the Sunda islands and independently devised a theory of evolution very similar to Darwin's. His book *The Malay Archipelago* is both a research report and an anthropological documentation of the life and customs of some of the tribes and peoples of the area. The naturalist's accounts of his encounters with the native people reveal certain aspects of how they thought.

Wallace mentions an encounter with members of the Dayak tribe from an inland region of Borneo. At that time, head hunting was still commonplace among tribes in Southeast Asia, but this did not prevent members of the tribes befriending Wallace. A common phenomenon even today in Southeast Asia, especially in Malaysia, Thailand and Indonesia, is for locals to respond affirmatively to questions to which they do not know the answer. Wallace remarked that it was difficult to obtain precise information or personal opinions from the Dayak. According to them, this was because if they told him that they did not know they might be lying! The crux of the matter lies in whether the person knows if they know or do not know something:

	Answer	Reflection	If I spoke ...
Question	I know	I know I know	...tell the truth
		I don't know I know	...I could lie
	I don't know	I know I don't know	...tell the truth
		I don't know I don't know	...I could lie

An exhaustive count (Indonesia)

Wallace dedicates a whole chapter to explaining how the Rajah of Lombok, the ruler of one of the islands of the Sunda archipelago, carried out his census of the population. From a mathematical perspective, a census consists of establishing a 1-1 correspondence between the natural numbers and the inhabitants of an area or region. In other words, counting them. It is not easy to put this problem into practice. The Rajah wanted to know exactly how many subjects he had under his domain. Furthermore, he did not want to know this statistically, but wanted them to be accounted for one by one. The importance of precision was because taxes were calculated per person. Nobody was to be exempt from paying and so the Rajah needed to know the exact number of people to charge.

The Rajah's solution needed a way in which people counted themselves and one that ensured the count was genuinely exhaustive. To do so, he made use of the cultural context, and herein lies a fundamental feature of the problem. It was impossible to force the people of each household to provide the answer. The census had to be carried out in such a way that nobody realised it was a census, and less still why it was being carried out. Only in this way would it be possible to ensure the information was reliable.

The Rajah summoned all the chiefs, priests and princes and told them that the great spirit of the volcano had appeared in his dreams. They must give the order to open pathways into the mountain to allow him to climb it and listen to what the great spirit was going to say to him. This is what he did, and the Rajah reached the top of the mountain with a great procession of dignitaries waiting for him below. After three days, the Rajah summoned the chiefs and priests again to tell them what the spirit had said.

According to the great spirit, terrible plagues and illnesses threatened the entire population of the island. It would only be possible to survive them by following his instructions. The great spirit ordered them to make twelve sacred *krises* (a krise is a dagger with a wavy blade that is common in Southeast Asia). To do so, each community from each district was to send a bundle of silver needles with one needle for each person in the area. When plague or illness struck a community, one of the twelve krises would be sent there, and if each house in that community had sent the correct number of needles, the plague or illness would cease immediately. On the other hand, if the number was incorrect, the sacred dagger would not work. This is what he did. And when a tragedy occurred in a community, he sent one of the

krises to eliminate it. If the danger passed, it was the virtue of the sacred dagger. If it persisted, it was due to the erroneous number of needles.

There is no doubt that if the count was exhaustive, it was thanks to manipulation of an ignorant and superstitious populace using indirect threats, topped off with attributing blame on the innocent. If things go well, it is thanks to the divinity; if they go badly, it is the fault of man, who in this case is guilty of an incorrect count.

The Kiowa (United States)

The Native American "Indians" became famous throughout the world thanks to Western films. The culture of the white man makes him appear master of the land on which he lives, and hence above nature, which he transforms at his whim. The world and the universe are, in a certain sense, at his service, and must respond to his desires. Indian cultures see things in a completely different light. The Indian perspective is that man belongs to the world and the land, and his relationship with nature must be based on a principle of equilibrium. Animals, landscapes, rivers and lakes, everything has a vital essence that must be respected. The natural elements are sacred and warrant the greatest respect.

Does this mean that the logic of the white settlers and the indigenous communities are different? It is possible that this is the case in certain aspects, but the contrast in philosophical approaches does not necessarily imply a change of logic. The following text is an adaptation of a Kiowa account of a peculiar character, whom we shall refer to as S, whose nature is to trick people:

S comes up against a stranger X. The stranger says to S:

— I don't know you. But I have heard people speak about you. You are the person who tricks everybody.

— Yes, I am. But I have left my medicine in the house and I can't trick you.

— What do you mean? If you are the person who tricks everybody, you will be able to do it without your medicine.

— No, I can't trick you without my medicine. If I had it, I would trick you. If you want, lend me your horse so I can go and get it and I will come back and trick you.

— Okay, I will lend you it. But you must come back with your medicine.

S gets on X's horse and while he is moving away, he brings it to a halt surrepti-

tiously. He turns to X once again:

– The horse does not want to go, perhaps it is scared of me. Lend me your hat.

X lends him his hat, but the horse stops again. Then S says to X:

– I'm scared of your horse. Give me your coat.

The same thing happens again, S asks for his cloak, and then his whip. As he moves away, S turns to X and says:

– Now that I have all your things, I have tricked you. I don't need any medicine.

The tale is a lesson in logic. Some expressions can be analysed from the formal perspective. We begin by defining what we mean by a trickster. If a liar is a person who never tells the truth, a trickster is someone who sometimes tells the truth and on other occasions does not. S tells the truth when he agrees that he tricks everybody, but he lies when he says he requires a medicine to do so and that he has left this at home.

Does this contradict what S goes on to say, that without medicine he cannot trick people? This is a logical implication:

p: without medicine $\Rightarrow$ q: I can't trick people.

Preparing the truth table for this logical implication, we see it is always true, except when the premise is true (1) and the conclusion is false (0):

p	q	$p \Rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

X, who is talking to S, appears to realise this when he replies that if the latter is the person who tricks everybody, he will not need any medicine to do so, which means the implication expressed by S is false. Herein lies the key to the tale and its logic. However, S insists that he cannot trick people without his medicine. X's naivety gives rise to the events that follow.

Kinship

Symmetry does not only appear, or is perceived, in the visual field. It is also implicit in the coexistence of members of a community, especially the relationships of kinship, be they in terms of blood or be they political. The equality of members cannot be understood without a symmetry in the relationship. The absence of symmetry in the relationship between parents and their offspring determines the social inequality between one and another. If A is the father or mother of B, then B is *not* the father or mother of A. The same does not hold for siblings. If X is the brother or sister of Y, then Y *is* the brother or sister of X. Siblings are part of the same generation and deserve – or at least it is assumed this is the case – equal treatment by their parents and society at large, both in terms of affection (respect, support, shelter, food) and socially and politically (education, and legal rights and responsibilities).

In the academic mathematics of the Western world, relations are studied because they can be used to define classes. The elements of a class are characterised precisely by sharing common features that determine the relationship. As an example, let us consider the relationship defined by the expression 'older than'. We say that an subject A is related to another B, written $A \sim B$ if 'A is older than B'. What properties does this relationship verify? To begin with, is subject A related to itself? That is:

$$A \sim A?$$

No, because something cannot be older than itself. The relation is not reflexive. If subject A is related to another B, is B related to A? In other words:

$$A \sim B \Rightarrow B \sim A?$$

This doesn't hold either, since if 'A is older than B', it is impossible for 'B to be older than A'. Hence, nor is the relation symmetric. And if a subject A is related to another B, which in turn is related to another C, what can we say about the relation between the first and the third? Are they related? That is:

$$A \sim B \text{ and } B \sim C \Rightarrow A \sim C?$$

In this case, the answer is yes, since 'if A is older than B' and 'B is older than C', then 'A is older than C'. This means the relation is transitive. We can conclude that the relation 'older than' is not reflexive or symmetric, but is transitive.

An example of a relation that is reflective, symmetric and transitive is 'being the same age as'. It is clearly reflexive, since one is the same age as oneself. It is symmetric,

since if A is the same age as B, B must be the same age as A. Furthermore, it is also transitive: if A is the same age as B and they are the same age as C, then A is the same age as C.

The majority of relationships for which these three properties hold (reflexive, symmetric and transitive) are equality relations. Hence the subjects or elements related by such a relationship form part of what is referred to as an equivalence class.

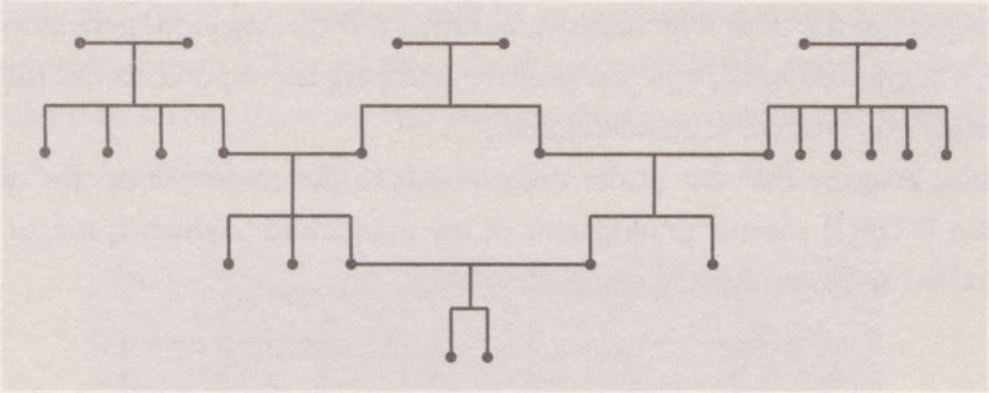
To begin with, it may seem strange to speak of equivalence classes, but people throughout the world do so on a daily basis – it's just that they don't use technical language, but everyday language. When we use the word 'apple', we are referring to a type of fruit, but we refer to it as an equivalence class within the class of fruits. If we specify a Golden Delicious apple, we are referring to an equivalence class within the class of apples. Being an apple and being a Golden Delicious apple are equivalence relations within the set of fruits, and apples, respectively.

Are there equivalence relations for kinship? The table below shows the properties of blood and political (shaded) kinship relations. It does not take account of gender. In this respect, being a brother or a sister is the same relation.

Kinship	Reflexive	Symmetric	Transitive
Parent of	No	No	No
Child of	No	No	No
Sibling of	No	Yes	Yes
Grandparent of	No	No	No
Grandchild of	No	No	No
Aunt/uncle of	No	No	No
Niece/nephew of	No	No	No
Cousin of	No	Yes	No
Spouse of	No	Yes	No
Parent in-law of	No	No	No
Child in-law of	No	No	No
Sibling in-law of	No	Yes	No

Since there is no relation for which all three properties hold, none of the relations is an equivalence relation. The closest we can find is 'being a sibling of', which is symmetric and transitive but not reflexive.

In our culture, the fundamental geometric model for kinship is the family tree. This represents the generational relationships of blood and marriage. In the following tree, marriages are shown by the horizontal connections:



The axis of the system of blood kinship is the generational line marked by the relationships of grandfathers, fathers, children and grandchildren corresponding to the vertical lines of the system. Blood relations at the level of each generation, or rather present in the horizontal lines of the diagram, correspond to siblings and cousins. The political relationships are those of spouses and siblings in-law.

The composition of the blood and social relationships creates other transverse relationships that make up the diagonals of the family tree. These are the relationships of aunt and uncle, niece and nephew, parents in-law, children in-law and grandparents in-law.

Gender aside, our system is a tandem one in the sense that the relations that are not symmetric (the majority) have a term that compliments them. This is not the case for siblings or cousins in the case of blood relations, or for spouses or sibling in-laws in terms of social relationship. If *A* is the sibling, cousin, spouse or sibling in-law of *B*, then *B* is a sibling, cousin, spouse or sibling in-law of *A*. This does not occur in anti-symmetric relationships:

Grandparent – Grandchild
Parent – Child
Parent in-law – Child in-law
Uncle/aunt – Nephew/niece

The family tree constitutes a geometric model of the relations of kinship as understood in the West. An algebraic model is also possible, and is shown in the following table. Let us construct an initial algebraic model of the relations of

blood descendants (excluding siblings, aunts/uncles, cousins and nieces/nephews) that correspond to five generations (grandparents, parents, ourselves, children and grandchildren) in a table. The numbers represent subjects of different, but successive, generations. The 0 corresponds to the origin, the generation of the person reading and interpreting the table. The negative numbers indicate previous generations (–1: parents; –2: grandparents), whereas positive numbers correspond to the following generations (1: children; 2: grandchildren).

Hence, imagine that the reader corresponds to generation 0. In this case, the operation (–1)★(1) means "grandparent of my grandchild", which is me, or rather 0. The table can be completed along these lines:

★	-2	-1	0	1	2
-2	-4	-3	-2	-1	0
-1	-3	-2	-1	0	1
0	-2	-1	0	1	2
1	-1	0	1	2	3
2	0	1	2	3	4

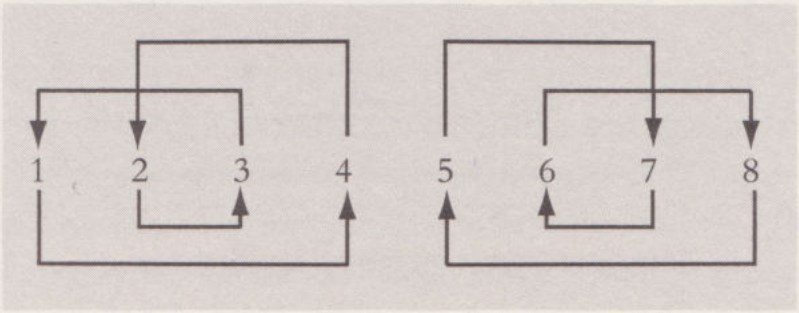
It should be noted that the operation ★, which is defined in the table, corresponds to the addition of the numbers. The composition of relations with themselves, represented by the symbol (∘), may give rise to others or remain the same. For example, the parent of a parent is a grandparent:

Parent ∘ parent = grandparent
Child ∘ child = grandchild
Sibling ∘ sibling = Sibling

The kinship system of the Warlpiri (Australia)

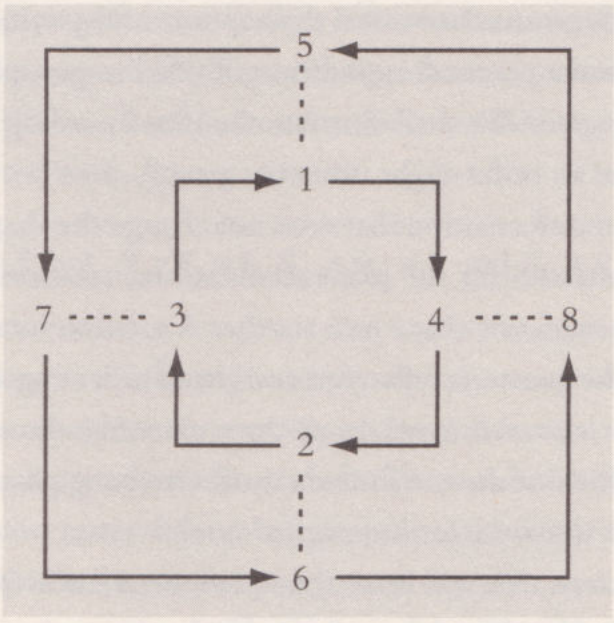
The Warlpiri are native to the Northern Territory of Australia and have an extremely complex system of kinship. The system determines the way in which they behave, are related and organise themselves socially and politically. It also determines the organisation and execution of their rituals. Just like for other peoples throughout the world, for the Warlpiri everything that exists is connected and forms part of a system of life established by their mythological ancestors who gave order to the world and made the mountains, rivers, flora and fauna, and gave these their names. Their ancestors also determined what was sacred and the rites and ceremonies that must be carried out.

The Warlpiri system of kinship is based on eight sections and is governed by a series of rules. Everyone belongs to one of the sections. A couple's children will be in a different section from their parents, depending on the mother. Representing the different sections using the numbers 1, 2,..., 8, the daughter of a woman from section 4 will be in section 2; her daughter will be in section 3; and the daughter of the latter will be in section 1. The relations for the sections 5, 6, 7 and 8 are established in a similar way. Hence there are two non-overlapping matrilinear cycles of order four: {1, 4, 2, 3} and {5, 7, 6, 8}.



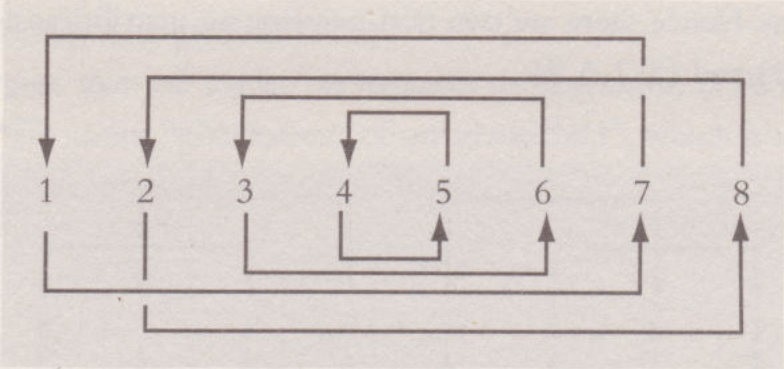
The matrilinear cycles in the kinship system of the Warlpiri people in Australia.

Another rule is that marriages are not permitted within the same section. The marriages are represented using broken lines in the following geometric model of the system:



Marriages in the Warlpiri kinship system.

Since the male sections are derived from the female ones, if a man from section 1 marries a woman from section 5, their child will belong to section 7. If the child then marries a woman from section 3 their child will be in section 1. Hence, we have returned to the same section. There are four patrilinear cycles of order two: $\{1,7\}, \{2,8\}, \{3,6\}$ and $\{4,5\}$:



Patrilinear cycles in the Warlpiri kinship system.

Hence, we have two matrilinear cycles of order four and four patrilinear cycles of order two covering the eight sections of the system. The complexity of the system does not stop here. The eight sections can be grouped in different ways to form sets whose relationships determine relevant social matters. For example, the groupings that correspond to hereditary rights are different to those of legal marriages or the associations for carrying out a task.

A formal and Western mathematical description of the system would state that it is nothing more than a practical application of what in group theory is known as an *order-eight isometry group*. We shall illustrate the idea by seeing how the isometries of a square constitute an order-eight isometry group.

An isometry is a transformation that does not change the shape or size of objects. There are three isometries on the plane: translations, rotations and reflections. A translation simply changes one shape into another. A rotation rotates a shape around a point referred to as the centre. A reflection consists of reflecting a figure with respect to a line segment (or virtual mirror). In all three cases, the dimensions of the shape remain the same. Which of these transformations can be applied to a square so that the resulting shape is the same as the original one?

The smallest rotation that will leave the square invariant is 90°. This is an order-four rotation, and if it is repeated four times, the shape returns to its initial position. If we represent this using the letter I (identity), the four rotations are $G4_1$, $G4_2$, $G4_3$

GEOMETRY IN THE MEASUREMENT OF TIME AND SPACE

We are always aware of the passing of time, so it might sound odd to ask whether time exists. Today we measure time in seconds, minutes, hours, days, months, years and multiples and divisors of these units. Not so long ago, distances were also measured by the time required to cover them. Some sailors built devices to measure units of time shorter than a day, a morning or an afternoon. One of these devices took the form of an empty coconut that had been cut into sections and had a small hole in the bottom. Setting it afloat in a tank of water, the water gradually flooded in through hole, until it filled the coconut making it sink. The process lasted about an hour.

Another of the systems is still in use – the sand glass. In its ideal version, the grains of sand would fall one by one through the narrow channel separating the two glass cones. This leads us to think of time as a discrete quantity that can be computed grain by grain. However, we have a continuous perception of time that is closer to the rotations of a radius around the centre of its circle. The measurement of time is closely related to the circle as are its sexagesimal angular division. The system has been inherited from Mesopotamian cultures and is also used for spatial orientation.

and $G4_4 = I$. The square also remains invariant when the following reflections or mirror symmetries are applied: (a) vertical; (b) horizontal; (c) diagonally ascending; and (d) diagonally descending. Let's now note that each of these reflections is of order two, since applying them twice in a row returns us to the point at which we started. If we use the letter S to refer to these mirror symmetries, we have: S_H, S_V, S_{D1} and S_{D2}. Since all these have order-two, the composition of each gives the same proportion we started with, I:

$$S_H \circ S_H = I,\ S_V \circ S_V = I,\ S_{D1} \circ S_{D1} = I \text{ and } S_{D2} \circ S_{D2} = I.$$

Not only are they not infinite, but they do not escape from the order-eight system. Let us now consider the correspondence between the Warlpiri kinship system and the isometry group outlined above. The two order-four matrilinear cycles correspond to the two order-four rotations, whereas the four order-two patrilinear cycles correspond to the four order-two mirror symmetries.

The Warlpiri may be unaware that their system of kinship corresponds to what is referred to as the order-eight isometry group in Western mathematics. However,

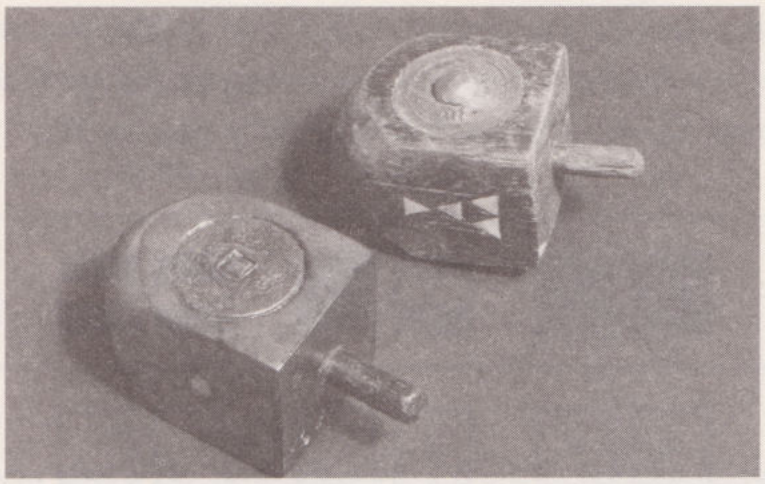
they have created such a system and their lives and social relationships are based on it. Thus do they conceive of and organise their social, political, religious and kinship relations. From a realistic perspective, their system is not a practical application of Western mathematics. They had a system of isometry long before relationships were classified in this way in the West. Their system is not only bound up with their culture, but defines it.

Equal bets

Betting games are common to all cultures and constitute a type of social relationship. Bets are made on one of the many possible results of a phenomenon, the occurrence of which, at least partly, is subject to chance, or rather the uncertainty means it is not possible to predict what will really happen. This is the case with horse racing, dice games and numerous betting games. By taking part in a game, a player declares that they know its limits and rules and, furthermore, accepts its random nature. In fact, without a margin for chance it would be impossible for games to really exist. Large sums are won when players bet on unlikely results, both probabilistically and mathematically, or in a social sense (nobody or almost nobody opts for that possibility).

Is chance understood in the same way everywhere? This question is difficult to answer. In some cultures, chance may be in the hands of the gods and thus constitutes a mode of expression. Devotees will consult oracles, cast stones or bones, or interpret the appearance of the entrails of an animal. In other cultures, the matter may be resolved by quantifying the possibilities of the results determined by the character or shape of the elements involved in the phenomenon, such as in the lottery and dice games. At any rate, betting games go beyond a culture's predominant determinism or indeterminism, since these types of games and conventions arise in practically all cultures.

The following photograph shows a pair of dice from the island of Lombok, in Indonesia. They do not have six faces. In fact they are spinning tops onto which four faces have been carved to make a die. When they are spun, they come to rest on one of the four faces. Not all the four faces are different, however. In one pair of opposite faces a coin has been embedded, whereas the other two faces have nacre embedded. When a die is thrown, two results are possible; let us refer to these as nacre (N) or coin (C).



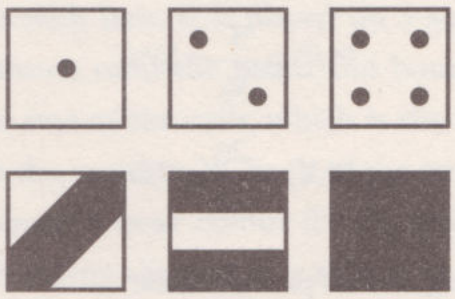
Dice from Lombok, Indonesia (photograph: MAP).

On one of the dice, the *C* faces have an extra piece of copper embedded, causing a bulge. Are the possible results equiprobable? Analysing the geometry of the dice, we may venture to state that this is not the case. Perhaps these faces weigh more than the others and give the object a shape that is elongated, not cubic. However, the definitive answer can be obtained by repeatedly throwing the pieces and observing the outcome. Out of 20 spins, *C* only appears twice.

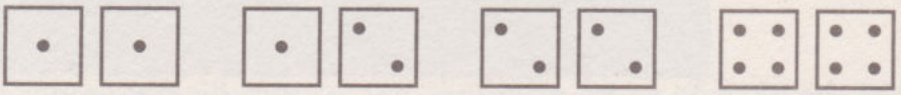
Whoever bets in favour of *C* will only do so a few times. After throwing the die a few times, they will realise it does not meet the fundamental requirement of the game. The balanced probability of the results. To play or bet with dice of this nature is meaningless, since we are already 80% sure of the outcome.

Daddu (Indonesia and Malaysia)

Daddu is a dice and betting game played in Indonesia and Malaysia, where it is known as selebor. It is played using two equal dice the six faces of which are painted as follows:



The game is played by four players, *A*, *B*, *C* and *D*. The dice are passed from player to player in a clockwise direction. Depending on the result, the throw may be *ottong* (winning: *G*), *mate* (losing: *P*) or *elang* (keep playing: *X*):

Winning throws (*G*)	
Losing throws (*P*)	
In other cases, the player has neither won nor lost (*X*)	

The game begins with player *A* throwing the dice. If *A* wins (*G*), they throw the dice again. If *A* does not win, or rather if they lose (*P*) or neither win nor lose (*X*), the dice are passed to *B*, who then throws them. If *B* wins (*G*), *A* loses; if *B* loses (*P*), *A* wins (*G*). And if *B* neither wins nor loses (*X*), the dice are returned to player *A*, and the game continues until one of the two, *A* or *B*, loses. At this point *C* enters the game and plays the winner. Once the game with *C* has finished, the winner plays *D*. And so on. There is no end to the game; the end is determined by the limitations imposed by the players. Hence, the losers are not excluded and can re-enter the circle. The game is played using bets, which are generally of the same value.

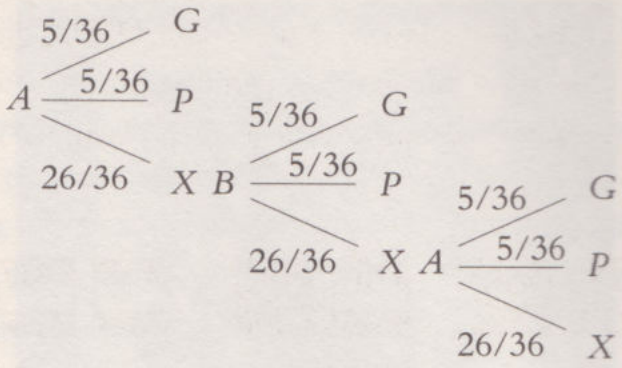
Hence, the probabilities of the first player winning (*G*), losing (*P*) or having to pass the dice on are as follows:

$$P(G) = \frac{5}{36} \approx 14\%.$$

$$P(P) = \frac{5}{36} \approx 14\%.$$

$$P(X) = \frac{26}{36} \approx 72\%.$$

This gives the following tree of probabilities:



It is a game in which the probability of A tends to stabilise around 50% as the game goes on. This means the relationship between the three probabilities is of fundamental importance:

$$P(G)=\frac{5}{36}=P(P)\Rightarrow p=q.$$

$$P(X)=\frac{26}{36}\Rightarrow r=1-2p.$$

The probability that A wins grows closer to 50% the longer the game is played:

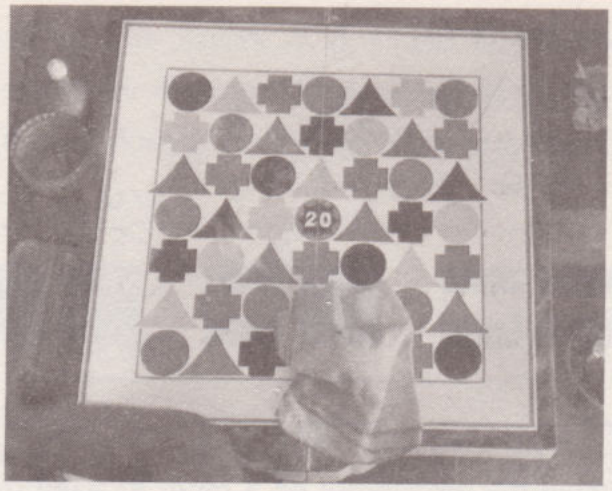
$$P(A=G)=\frac{5}{36}+\frac{26}{36}\cdot\left(\frac{5}{36}+\frac{26}{36}\cdot\left(\frac{5}{36}+(\ldots)\right)\right)=$$

$$=\frac{5}{36}+\frac{5}{36}\cdot\frac{26}{36}+\frac{5}{36}\cdot\left(\frac{26}{36}\right)^2+\frac{5}{36}\cdot\left(\frac{26}{36}\right)^3+\ldots=$$

$$=\frac{5}{36}\cdot\left[1+\frac{26}{36}+\left(\frac{26}{36}\right)^2+\left(\frac{26}{36}\right)^3+\ldots\right]=\frac{5}{36}\cdot\sum_{k=0}^{n}\left(\frac{26}{36}\right)^k\xrightarrow[n\to\infty]{}\frac{1}{2}.$$

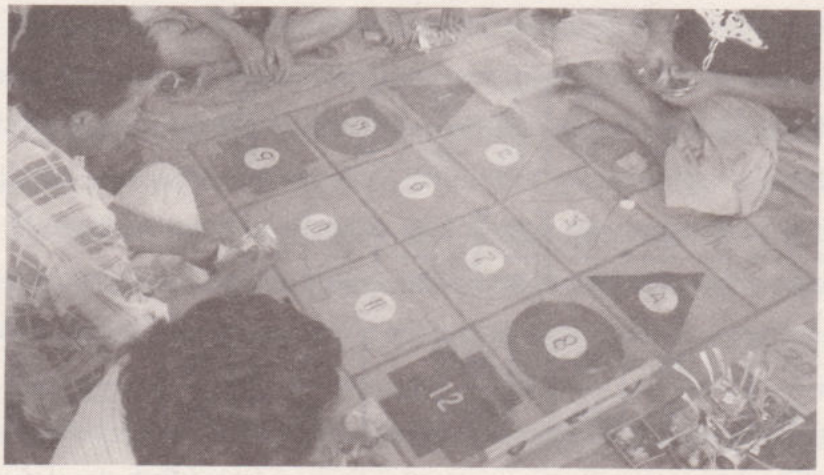
Bola adil (Bali)

This is a game of chance with bets. It is played on a square 7×7 board with 49 concave cells. A ball is thrown onto the board and bounces off its edges until it stops and comes to rest in one of the cells, which is the winning cell. The cell in the centre is marked with the number 20. Each of the remaining 48 cells contains a shape (circle, triangle, cross) whose colour (black, yellow, green or red) varies depending on a pattern based on diagonals, as shown in the photograph below:



Bola Adil board (photograph: MAP).

Each shape is repeated four times in the same colour for each of the four colours. Hence, of the 48 cells, there are 16 circles (4 black, 4 red, 4 yellow and 4 green), 16 triangles and 16 crosses. The bets are placed on an additional 3×4 board with 12 cells numbered from 1 to 12, as shown in the photograph below:



Bola Adil betting board (author's photograph).

The money bet on the winning cell is multiplied by 10. Bets can be placed on one or more cells. The prize obtained is multiplied by 10 times the part assigned to the option corresponding to the cell in which the ball has come to a rest. For example, imagine a player bets 30,000 rupees, split between the cells 4 (black triangle) and 8 (black circle). If the ball comes to rest in a cell with a black circle, the player will win 150,000 rupees, ten times the value of the bet corresponding to the shape (15,000 rupees). The probability for each cell is:

$$P = \frac{1}{49} = 2.04\%.$$

If the ball comes to a rest on the cell in the centre (marked with the number 20) all the money that has been bet goes to the bank. This is a possible result that is not available to the player when making their bets, since they do so on a board with options from 1 to 12. From their perspective, and given that bets are placed on 12 cells, the probability of winning would appear to be:

$$P = \frac{1}{12} = 8.33\%.$$

However, the real probability is lower, since the board for the bets does not include the possibility of the bank winning.

$$P = \frac{4}{49} = 8.16\%.$$

Examining the board for placing the bets, we can ask which is more favourable: betting on two horizontal numbers or two vertical numbers? For example, what is more probable, the combination 1-2 or the combination 1-5? The combination 1-2 wins if a triangle comes up, regardless of whether it is red or green. The combination 1-5 wins if a triangle or circle comes up, but only if they are red. However, since there are as many red triangles as green ones (four of each), which is the same as the number of black circles and black triangles (four of each), the probability is the same:

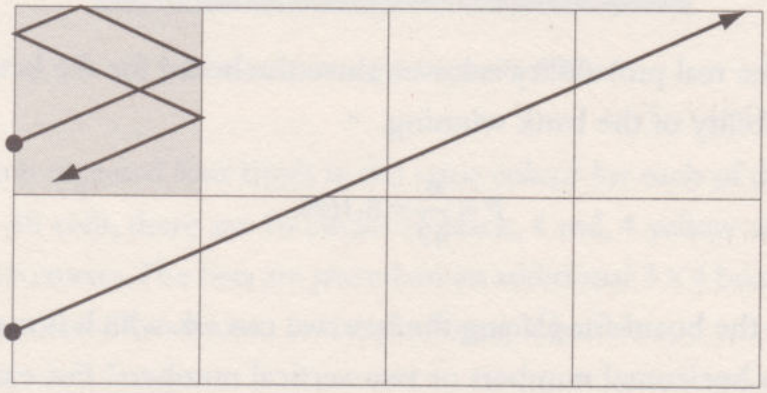
$$P(1.2) = P(1.5) = \frac{8}{49} = 16.3\%.$$

The fact that the players are aware that betting on a single result is too risky is shown by the fact that the most common bet is on two numbers between 1 and 12 on the board used for placing the bets.

Two relevant issues related to this game are related to the board onto which the ball is thrown. The first is related to its shape: why is it square? The other is related to the number of cells: why is a 7×7 board used? Why not use a rectangular, triangular, hexagonal or circular board? Would it be possible to play the game on boards with 25, 36 or 100 cells?

The shape of the board affects the path taken by the ball, which is determined by the direction in which it is thrown and how it bounces off the sides. The shape is a geometric matter. In theory, certain throws may be regarded as less random,

such as those in which the ball follows a path on the board determined by the midpoints of its sides. This would be the theoretical result if the ball was released from an arbitrary point on one side of the board at an angle of 45°. However, this is purely theoretical, since the concave shape of the cells means that every time the ball passes over one of the cells without passing exactly over its centre, its direction is affected, giving it the random property that justifies playing the game. Hence, paths as geometrically predictable as the following polygonal path marked in black on a light grey board never arise:



A purely mathematical model of the ball's path is impossible, since it would need to take into account physical factors such as friction and the forces derived from the uneven, concave shapes of the cells, which may cause the directions to change upon entry and exit. The number of variables that need to be taken into account makes the problem extremely complex. To understand this is to understand the origin of the random nature of the game and the bets.

The number of cells on the board is a numeric matter. Given that we have three shapes and four colours that can be combined to give 12 possibilities in addition to an extra possibility whereby the bank wins all the money, the number of cells C must be one more than a multiple of 12:

$$C = 12 \cdot k + 1, k \in N.$$

Bearing in mind its square shape, C must also be the square of a natural number. This is achieved by taking the squares of the multiples of six, adding or subtracting a unit:

$$(6 \cdot \lambda \pm 1)^2 = 36\lambda^2 \pm 12\lambda + 1 = 12\lambda \cdot (3\lambda \pm 1) + 1 = 1 + \text{multiple of } 12.$$

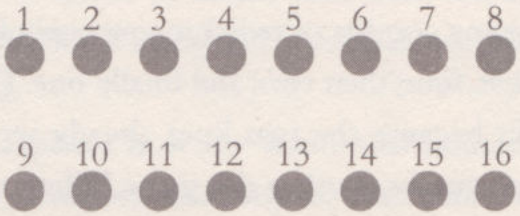
Hence, the table could also have a different number of cells, although this would make the probabilities too small ($C > 49$) or too large ($C = 25$):

n	$(6n+1)^2$	$(6n-1)^2$
0	1	1
1	49	25
2	169	121
3	361	289

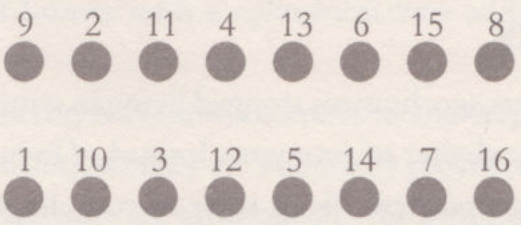
A kpelle game (West Africa)

In her book *Africa Counts*, Claudia Zaslavsky describes a game played by the Kpelle. The game consists of arranging 16 pebbles in two lines of eight. A stone is chosen and another person who has not seen the selection has to guess which of the pebbles has been chosen. To do so, they may ask which of the two lines contains the stone that has been selected up to four times. After each answer, the person can reorganise the stones into two lines.

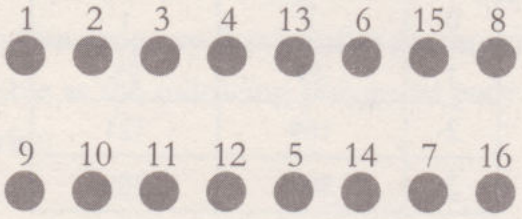
All the stones must be the same and indistinguishable from each other. They may, however, be different colours to help follow their movement.



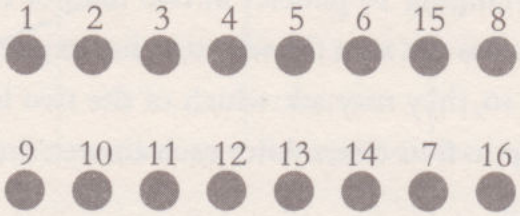
The solution to the problem lies in how the pebbles are redistributed upon each answer to the questions that are asked. Imagine that stone number 13 has been chosen, but that we do not know this. Hence, when we see the two lines, we will ask "Which one contains the stone that has been chosen". The answer will be that it is in the bottom line. We will then switch the odd stones of both lines:



When we repeat the question, we are told that the chosen stone is now in the top line. Since its position has changed, we know it belongs to the group {9, 11, 13, 15}. We can now switch half of these, for example 9 with 1 and 11 with 3:



This time, we are told that the stone is still in the first line. Hence it must be 13 or 15. We must now switch just one of the stones. For example, 13 and 5:



The final answer will reveal which stone has been chosen, since it has returned to the second line. Hence, the number is 13.

The strategy of the game consists of switching precisely half of the stones as the answers are provided. First, four; then two; and finally one. The fourth answer gives the solution. This works because the two lines already separate the eight stones that represent half the pieces involved in the game. When we are told which row contains the chosen stone, half the stones are now excluded. Hence, if our strategy guarantees that each answer refers to half of the previous stones, we will inevitably reach a unique conclusion because:

$$\frac{16}{2}=8 \rightarrow \frac{8}{2}=4 \rightarrow \frac{4}{2}=2 \rightarrow \frac{2}{2}=1.$$

Inhabiting geometry

Tens of thousands of years ago, humans stopped living in natural shelters and decided to seek protection in the shelter of geometry. Instead of living in caves, for example, humans transformed natural items from their surroundings to build houses. This

involved establishing a lasting order and shape, which has been developed with the passing of time.

The majority of modern dwellings are based on polyhedra, and the majority of these are rectangular prisms. Tens or hundreds of families are grouped together in colossal hexahedra packed into cities throughout the world. Humans also live in, or have lived in until very recently, shapes derived from or inspired by the circle, such as cylinders, columns and even spheres. The fundamental aspect of an inhabitable hexahedron is the right angle. The walls of these houses are perpendicular to the ground and to each other. The rooms and spaces of a house also replicate this model. The majority of the furniture which surrounds us also has this shape. Many tables, chairs, shelves, wardrobes and beds are designed in the shape of a hexahedron, meaning that they perfectly fit the floor and walls where they are placed. Other items, such as lamps, can be designed with more freedom.

In addition to the individual aspect of dwellings, peoples and cultures are also characterised by the way in which they are combined to create communities. There are rectangular and circular layouts, but there are also layouts that are not based on a specific shape, having been developed without a predetermined pattern.

We can find examples of circular dwellings throughout the world: for example, the conical trulli of Alberobello in south-east Italy; the huts of many African peoples; the teepees of the Native Americans; and the houses of the Cumbi of Indonesia and Atoni in Timor. The igloos of Inuits, built from ice, are hemispherical. Other dwellings combine a cylindrical shape with a conical roof, something that is common in many parts of Africa.

Claudia Zalavsky explains how the traditional houses of the Chagga people, who live on the slopes of Mount Kilimanjaro, were built. The first step was to fetch the tallest man in the community. He then lay down on the floor with his arms outstretched. The radius of the house would be between two and three times the man's arm span. This length would be used for a string that was attached to a stake. The circumference was marked on the ground by walking full circle around this post. The height of the door was given by the extent of the man, and the width by the perimeter of his head measured using a piece of string. On the other hand, the Kikuyu in Kenya built houses with a cylindrical base and a conical roof that was covered with leaves.

In spite of the fact that it is common to refer to the teepees of North American Indians as conical constructions, they are essentially polyhedral. In fact, their shape is pyramidal. A series of long posts are positioned in the ground in a circle (determining

the vertices of a polygon that is quite regular) pointing up into the air. These posts form the edges of the pyramid that is made by the skins that cover them. Teepees can be easily taken down and moved if required.

The teepee is a traditional dwelling of some North American native cultures.

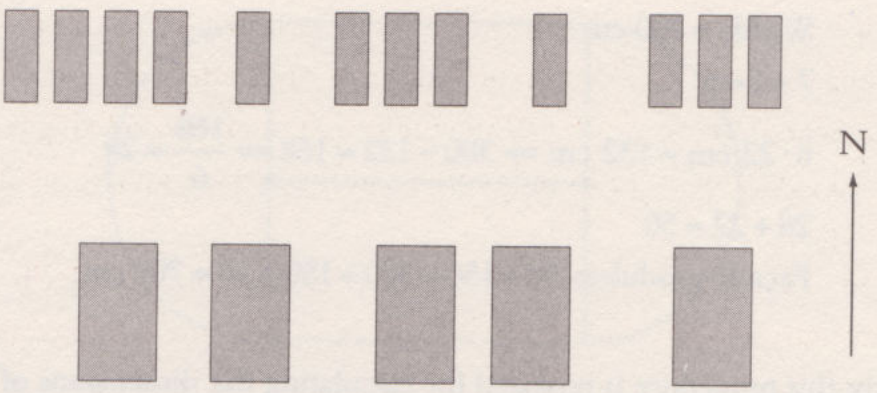
In fact, it is the roof that gives the house its conical shape. The traditional houses on the islands of Flores and Timor, in Indonesia, are perfect cones since the roof practically descends to the ground. While the structure of this covering is pyramidal, the leaf-based covering smooths the profile and the surface, giving the dwelling its definitive aspect.

African cultures often create their settlements and communities by grouping together houses depending on their shape. Those that are rectangular are often grouped into oblong-shaped districts; those that are circular are grouped together in a circular or elliptical pattern.

Some of the traditional African houses have ornamentation on their door frames and inner walls. Teepees also have symbols and designs on the hides to identify the occupants. The ice does not facilitate such things. Igloos are built using blocks of ice that have been moulded to create spherical spaces. However, the dome is erected in a helical shape, with the radius of the curve closing as it ascends. The dome of the igloo is closed using larger blocks than those used at the start of the construction.

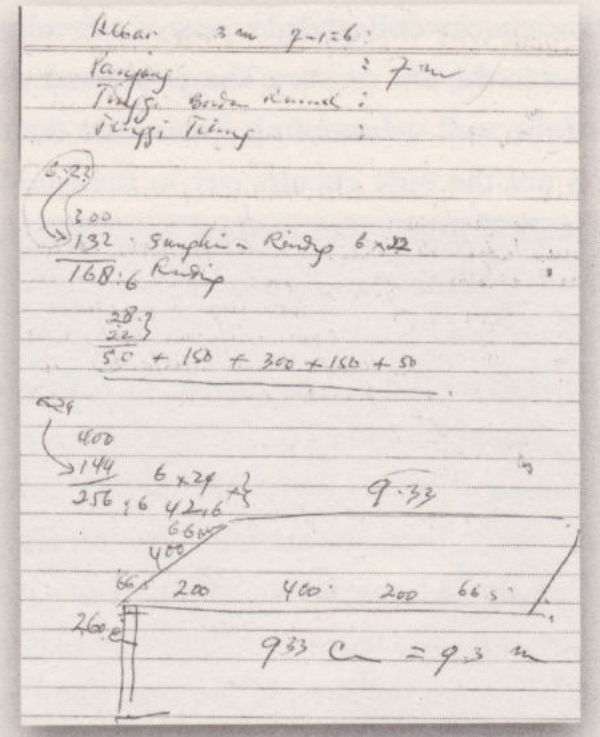
The layout of the ancient city of Baghdad is perfectly circular. Khalifa Al-Mansour ordered its construction in the 8th century. The palace and the mosque are in the centre. The double adobe wall surrounding it has four doors that face the four cardinal points. This is not the only circular city in the Middle East. Al-Mansour possibly took his inspiration from other circular cities that had been constructed previously, such as the city of Gor (now Firuzabad in Iran), founded by the Sassanid king Ardashir I during the 2nd century.

A different case is that of the Toraja people in Sulawesi, Indonesia. Their traditional houses are rectangular and characterised by three clearly differentiated levels. However, their unique character is given by the saddle-shaped roof. However, the fundamental aspect of the Toraja house is its position and its status as a family, social and cultural hub. A Toraja house is much more than a living space to provide shelter. Traditionally, Toraja houses face north, meaning that in Toraja settlements, houses are aligned in batteries, one beside the other and in parallel, all facing north. In front of each house, there are rice stores (one or more). Since the houses point towards the north, the stores point south, face-to-face with each house. The space left in the middle is where ceremonies and rituals take place. Each family is associated with the family house, which constitutes its point for meetings and assemblies and where the remains of its dead are placed until they are finally buried.



Structure of a Toraja settlement (Sulawesi, Indonesia).

The dimensions of the traditional Toraja houses and granaries are determined in advance and their proportionality obeys the ratio 7:3. The builder Marheen Madoi provides a written explanation of how these features are determined:



Hand-written explanation of the dimensions of a traditional Toraja house.

The interpretation of this explanation is somewhat clearer. If we incorporate certain facts that are not explicit in the builder's document, which skips several logical steps, we get:

Width = 300 cm

$7 - 1 = 6$

$6 \cdot 22 \text{ cm} = 132 \text{ cm} \Rightarrow 300 - 132 = 168 \Rightarrow \frac{168}{6} = 28$

$28 + 22 = 50$

Façade modules: $50 + 150 + 300 + 150 + 50 = 700$ cm.

Similarly, this procedure is repeated for calculating the dimensions of a building with a width of 4 m, although this time a value of 24 cm is used instead of 22 cm.

Width = 400 cm

$6 \cdot 24 \text{ cm} = 144 \text{ cm} \Rightarrow 400 - 144 = 256 \Rightarrow \frac{256}{6} = 42.6$

$42.6 + 24 = 66.5$ *(sic)*

Façade modules: $66.5 + 200 + 400 + 200 + 66.5 = 933$ cm.

A full explanation begins with the idea that both the house and the granary have rectangular footprints with dimensions that conform to the ratio 7:3. This rectangle is arranged in a grid of 14×6 modules. The 14 units of the longest walls are grouped into $1+3+6+3+1$. If the width of the building is to be 3 m, its length must be 7 m:

$$\frac{x}{300 \text{ cm}} = \frac{14}{6} \Rightarrow x = 700 \text{ cm}.$$

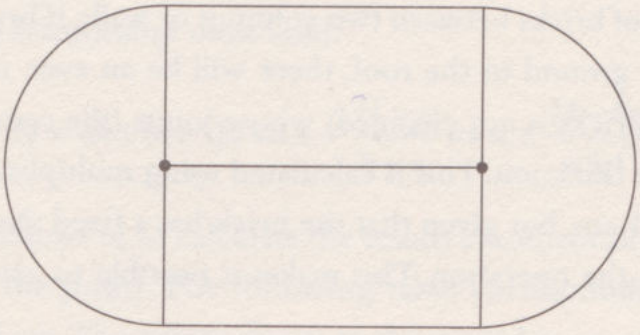
This means each cell is a 50-cm square and the 14 units of the sections of the two longest walls will have lengths:

$$50+150+300+150+50.$$

The same holds for a width of 4 m. Hence, the total length will be 9.33 m and the modules will be distributed as follows:

$$66.6+200+400+200+66.6.$$

The Shuar people inhabit the the Amazon rainforest in south-eastern Ecuador. One of their peculiar trademarks is their round houses. Although their bases are square, a semicircular addition to two of their opposite sides gives them an elongated appearance, as shown in the following image. The height is the same as that of the ridge, the horizontal post that serves as the axis for the roof.



However, the Shuar house is much more than a place to provide shelter from the rain or to store belongings and utensils. Like the Indonesian Toraja houses on the other side of the world, it is a scale reproduction of the cosmos, a representation of the universe. The space inside is divided according to genders and their roles in society – according to the beliefs of the Shuar. It also reveals the role that must be played by each member of the family in the community. In this respect, in addition to having a clear practical purpose, the main post that holds up the roof constitutes

an expression of the link between the earth and the sky, the world below and the world above. Shuar celebrations take place round this post.

Technology and mathematical thought

Nowadays, in most of the world work is based around the same tool: the computer. The difference lies in the software that is used, since each profession requires different and often specialised computer programs. The use of computers has become almost essential, to such an extent that many users have learnt how to modify them so they can use them autonomously. There are people who have even developed sub-routines and small programs in order to streamline calculating tasks.

Most professional people are able to use Excel spreadsheets. There is no industry in which it is not necessary to prepare accounts, invoices, finalise balances or calculate the terms of a relationship based on quantities. For many professionals working with spreadsheets is a way of rediscovering mathematics many years after having stopped studying it – studies in which they wouldn't even have seen a computer. The world of design and food are good examples of where mathematical activities of this nature take place.

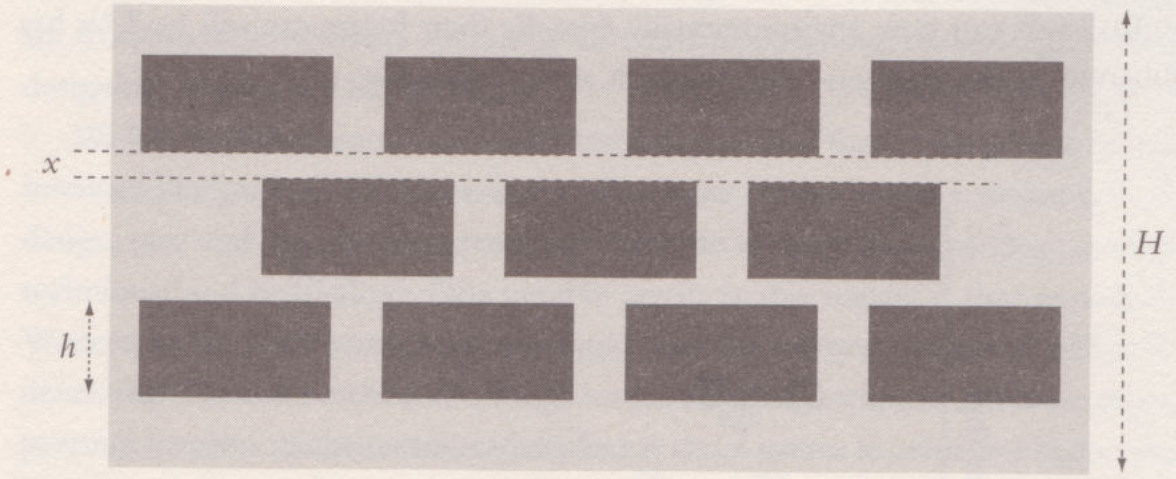
The distribution of courses in construction

A course is a series of bricks between two columns or walls. If bricks are distributed correctly, from the ground to the roof, there will be an even number of courses (the height of the bricks is not changed) whose joints (the cement joining them) will be of the same thickness. This is calculated using multiplication and division. The joint is often 1 cm, but given that the brick has a fixed shape, the joint is the flexible variable in the operation. This makes it possible to gain or lose 1 mm as required.

The following technique is used in practice: the measurements of the height of the brick (h) and the joint (j) are taken and a mark is made on a plank of wood at a distance of $d = h + j$ from one of its ends. Sequential marks are than made for the following values, which are obtained using a calculator: $[d] + d$, $[d + d] + d$, $[d + d + d] + d$, ... The plank of wood marked with these equidistant markings is the gauge used for the courses. It is done in this way to avoid having to measure the distance d for each course. If the result of the calculations is of the form 5.8 cm, it would make it extremely complicated for bricklayers to add this number using a

measuring tape. It is much easier to calculate it once and transpose it sequentially and automatically onto the plank of wood.

The situation is represented in the following illustration. The initial values are: H (height of the space), h (height of the brick), x (height of the joint) and n (the number of courses required to cover the space). The value of x is often around 1 cm, but as we have noted there is a certain margin of tolerance of approximately 1 mm.



Distribution of courses.

The following relationship must hold:

$$H = nh + (n+1)x \Leftrightarrow n = \frac{H-x}{h+x} \Leftrightarrow x = \frac{H-hn}{n+1}.$$

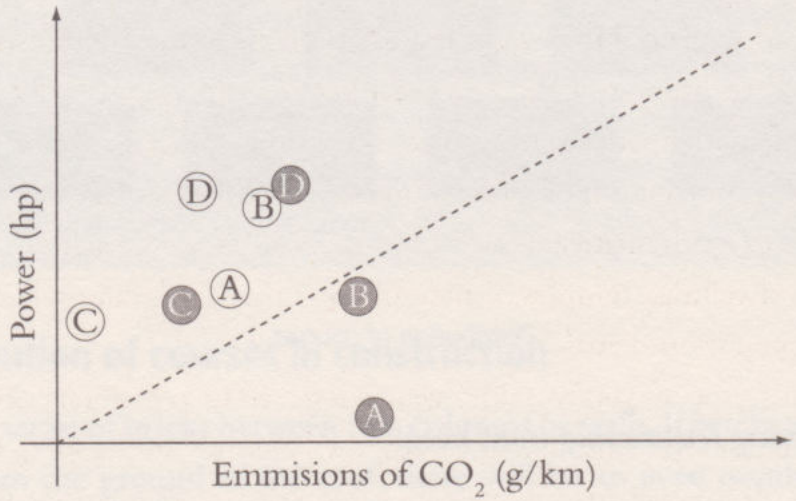
A spreadsheet allows us to calculate the results automatically (number of bricks and thickness of the joint). The following table corresponds to the case when $H = 3$ m and $h = 5$ cm. The values closest to those used in practice in construction are in bold.

H (cm)	h (cm)	n	x (cm)
300	5	51	0.87
300	5	**50**	**0.98**
300	5	**49**	**1.1**
300	5	48	1.22

New functions, new graphs

The problems of the modern world are not the same ones it faced centuries ago. A major concern today is the environment. Scientists have shown that if CO_2 emissions are not curbed, we will end up harming the planet. It is a problem that is difficult to solve, since the majority of the global economy is based on the use of fossil fuels.

Car manufacturers have become increasingly aware of this and modern cars are more environmentally friendly than they were, something that is stressed in car adverts. This is why car brochures often include graphs at the end to explain to the buyer just how environmentally friendly their future car will be. This has given rise to the creation of graphs such as the following:



Comparison of emissions/power of new models of cars (white circles) and older models (grey circles).

The best-case scenario is to manufacture high-power models with low emissions, the profile of which means they will be located towards the top left of the graph. The worst-case scenario is the opposite – the power of the car is low and it emits a high level of CO_2 into the atmosphere, situating it in the bottom right. In the situation shown in the graph, the new models are better than their predecessors since they combine more power and lower emissions (the cloud of white circles is above and to the left of the cloud of grey circles). Furthermore, each new model in particular represents an improvement over its predecessor (each letter in a white circle is to the left of or above the corresponding letter in the grey circle).

Epilogue

We began this book tens of thousands of years ago by analysing the geometric properties of a South African petroglyph. Since then, innumerable peoples and cultures have inhabited the planet, all characterised by their idea of the world and life, their way of doing things, their beliefs and rites and their architecture.

One of the aspects that is common to many cultures is an interest in doing things well and being able to reproduce them. It is not far-fetched to think that such activities involve mathematical knowledge. Indeed, Alan Bishop identified six mathematical activities shared by all cultures: counting, measuring, locating, designing, playing and explaining.

We have travelled around the world, touching on all of these, with greater or lesser intensity. The general conclusion is that all cultures count, calculate, measure, locate, design, play and explain. However, they often do so with different ideas, symbols, techniques and technology. One notable aspect in this respect is that outside the Western world, mathematics is not isolated from the cultural context in which it is developed. The construction of a Toraja house, a Buddhist stupa or an Aztec stepped pyramid involves mathematical activity, but is also a means to achieve a higher end. In addition to dwellings, temples or mausoleums, they have cultural and social roles that is their true raison d'être.

The study of a specific body of knowledge referred to as mathematics is a relatively recent idea that does not exist in many traditional cultures. In the West, a distinction is made between decorative arts, engineering architecture, and religious architecture. In other places, these classifications are not used. Selecting the mathematical knowledge implicit in a culture may be regarded as a sort of mutilation. From the perspective of vernacular culture, such a disassociation would lack meaning, since cultural expression is never one-dimensional.

Men and women from many peoples communicate with and show respect to their gods by means of prayers and offerings. This is why these rites follow a pattern, an order, and possess the rigour worthy of the deity. In Bali, offerings are placed in trays made from palm and banana leaves. They are simple materials that can be afforded by all, but they are also given a geometric uniformity. This is the responsibility of the women, and the know-how of making these receptacles is handed down from mother to daughter. A similar phenomenon occurs in the states of Kerala and Tamil Nadu, in the south of India, with colourful kolam patterns.

Numbers are counted and calculated everywhere, but context is everything and gives rise to indigenous arithmetic techniques for commercial activity. Salespeople in African markets and bus drivers in India have developed strategies for multiplication and division that do not require the use of a pen and paper. Some of these constitute applications of algebraic properties that have been learned from or are inspired by the academic world of mathematics, but others have developed independently.

Is there a culture that has not been interested in symmetry? Symmetry is a human feature. Perhaps this is why everything that is done by man, at least in a more traditional context, tends to be, or has been at some point, symmetric. Symmetry is to be found in dwellings throughout the world, in temples, city plans, ornamental designs, tools... We live in a world of symmetry from which even the most avant-garde schools of design can hardly escape. Symmetry is traditionally the face of beauty. From this perspective, if something is not symmetrical, it cannot be beautiful. Moreover, it is beautiful as a result of balance, since symmetry and balance are both intimately related. Hence why all peoples and cultures have used this relationship to develop the features and symbols that characterise them.

Other aspects from which no culture can escape are logic, games and betting. Kinship is a congenital logic that determines social relationships. For their part, games and bets are models of uncertain and invented worlds to which risk is of fundamental importance. The desire to win and the fear of losing are vital drivers. Chance is used to recreate these feelings in controlled situations. We do not know if chance exists or is a lack of knowledge, but betting games would be meaningless without this factor of uncertainty, which in the end is responsible for quantifying risk. The design of situations involving chance is linked to mathematics. Dice are cubes with equiprobable faces and the elements of the Bola Adil are purely geometric. Geometry, in the case of symmetry, contributes to the creation of random situations that give rise to the acceptance and understanding of chance among participants.

Observing all these activities taking place, it is hard not to think of the mathematical ideas required for them to exist. The interest in knowing them is the same one that motivates us to find out about the world. Why look for mathematical ideas outside our culture? Because, as we have seen, outside our environment different and enriching things occur. Waiters throughout the Malayan archipelago fold napkins by dividing the right-angled corner into three equal parts. But they do not do so following a geometric method from the academic mathematical context, but instead use a more effective and practical vernacular procedure.

Ethnomathematics allows us to discover people, cultures, techniques, tools and methods that enrich the mathematical knowledge of other cultures like our own. This enrichment does not only lie in new or different ideas, but in the new mathematical problems that are identified precisely as a result of cultural interaction.

How can we discover ethnomathematics? From a millennia-old petroglyph, it is possible to formulate hypotheses about the mathematical ideas that inspired it. It is impossible to confirm these hypotheses, since we cannot ask the creator, and nor do we have the tools used to carry out the work. The hypotheses regarding the mathematical knowledge required to create a cultural product such as a wooden sculpture or a textile garment are more plausible when we have the opportunity to observe how they are made. The methods and technology used and the language in which the creators refer to the things they make provide us with reliable clues to how they think.

However, it may be the case that, in spite of observing a process from up close, our mathematical model of what the person executing it is thinking is wrong. This was the case with the folding of napkins, because the actions observed during the process were indistinguishable from those required by the mathematical model prepared by the observer. The solution is to ask those doing the work. It is only then (and even still we may still have reservations) that we can be certain about what they are thinking.

There are animals that build architectural wonders. Bees, spiders, birds and dung beetles can create almost perfect balls, extremely regular geometric lattices and hexagonal cells. Looking at their work and observing their labours, we would say that their cells, webs, nests and dung balls are the fruit of mathematical ideas. It is possible, but there is a fundamental difference between these creatures and human beings: we cannot ask them and hence we can only make hypotheses about what they do.

Once we have identified mathematical knowledge, what can we do with it? The answer may lie in improving mathematics – both the vernacular one from which the knowledge has arisen and the foreign one that has identified it. Furthermore, we must do so in two directions: from the extra-academic context to the academic context and vice versa. Therein lies the importance of education. Belonging to a culture means appropriating its characteristic aspects, learning its language, its customs, its philosophy of life, is rites and beliefs, its methods of exchange, living in houses built using its architecture, eating its food, participating in its games and, of course, learning its mathematics. We have now seen that culture cannot exist without mathematics. Nor can we belong to a culture without learning its mathematics.

We live in an increasingly globalised world connected as a consequence of technology. The fact that technology would not exist without mathematics should not make us think that outside our extraordinarily technological world there is no mathematics from which we can learn. The universality of mathematics is not an *a priori* idea of Platonic scope, but a consequence of the ethnomathematical knowledge developed by all peoples and cultures. We have seen part of this in this mathematical odyssey that sadly must end here.

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All peoples and cultures have their own distinctive view of the world, influenced by beliefs, rites, customs, artistic endeavours, language, food, technology… and mathematics. This book takes us on a fascinating numerical journey through history and across cultures, stopping off throughout the length and breadth of the globe – with the mathematical imagination as our guide!